

Light paths of normal and phantom Einstein-Maxwell-dilaton black holes

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Abstract

Null geodesics of normal and phantom Einstein-Maxwell-dilaton black holes are determined analytically by the Weierstrass elliptic functions. The black hole parameters other than the mass enter, with the appropriate signs, the formula for the angle of deflection to the second order in the inverse of the impact parameter allowing for the identification of the nature of matter (phantom or normal). Observation of the so called relativistic images on photon spheres also allows for such a determination. Scattering experiences may favor black holes of Einstein-anti-Maxwell-dilatonic theory for their high relative discrepancy with respect to the Schwarzschild value. For the cases we restrict ourselves to, phantom black holes are characterized by the absence of many-world and two-world null geodesics.

PACS numbers: 04.20.Gz, 04.50.Gh, 04.70.-s

1 Introduction

Most experimental settings for testing gravitational theories are designed to evaluate trajectories of light rays. Accuracy in this field is a growing interest. From this point of view, the leading experimental settings are aiming to achieve high accuracies beyond the known first order level and to reach a sensitivity of 1 part in 10^9 in measuring the Eddington parameter γ [1], which is an important parameter in post-Newtonian formalism.

On the theoretical front, workers have been striving to evaluate exactly light paths using (hyper)-elliptic functions mainly the Weierstrass elliptic functions denoted by \wp [2, 3]. From the one hand, this has provided answers to some open questions, for instance, whether the cosmological constant could be a cause of the Pioneer anomaly [4], has raised the question to whether lensing could be used as a test of the Cosmic Censorship [5] and has lead to discover new light paths, the Pascal Limaçon trajectories for black holes with cosmological constant [6]. From the other hand, the analytical solutions derived so far, [7–23] to mention but a few, could be useful for any of the experimental settings aiming to test gravitational theories. Moreover, they provide new academic techniques for tackling motion of massive and massless particles in the geometries of various gravitational fields and may serve as references for testing the accuracy of numerical methods [24] and provide unique benchmarks for testing and improving perturbation and decomposition methods [25]. For that purpose it is very helpful to have relatively simple solutions.

In case of spherical symmetry, one of the equations governing geodesic motion reduces to

$$\left(\frac{dr}{d\phi}\right)^2 = P(r) \quad (1.1)$$

where $P(r)$ is a polynomial function of the radial variable r , the parameters of the solution and the constants of motion. Depending on the dimension of the space-time, $P(r)$ may be reduced, as described in [16], to a polynomial of degree 3 or 5. We are interested in the former case and we assume that (1.1) is brought to

$$\left(\frac{d\rho}{d\Theta}\right)^2 = 4\rho^3 - g_2\rho - g_3 \quad (1.2)$$

by coordinate transformations. Here g_2, g_3 depend on the parameters of the solution and the constants of motion. So far no special terminology has been introduced to simply notations and expressions. We introduce the following terminology to describe (1.2) and the related polynomial and coordinates. We shall call (1.2) Weierstrass differential equation¹, $w(\rho) = 4\rho^3 - g_2\rho - g_3$ Weierstrass polynomial and (ρ, Θ) Weierstrass coordinates.

We bring (1.1) to (1.2) by a series of coordinate transformations relating r to Weierstrass radial coordinate ρ where $\rho(r)$ is a non-trivial and nonlinear transformation; however, $\Theta(\phi)$ is a linear transformation and in many cases $\Theta = \phi$, where ϕ is the azimuthal angle.

Most workers prefer to use the effective potential approach by which they determine all planar trajectories (absorbed paths (captured photons), scattering paths, trapped or confined paths, (un)stable circular paths, spiral paths approaching the circular paths from above and/or below and some other special closed curves). The method we shall apply is entirely based on the properties of Weierstrass differential equation and of its polynomial. We shall develop and use this method, which has been used in [9, 19] (and partially used in [6, 8]), leading to a systematic approach for all problems governed by (1.2). This will allow us to determine all types of trajectories.

None of the papers mentioned above has ever dealt with light paths of normal black holes of Einstein-Maxwell-dilaton (EMD). One of the purposes of this paper is to address this question; the other one is to extend the analysis to that of light paths of phantom black holes of EMD and to draw a comparison between the systems of trajectories for a given ratio of charge to mass squared ($a^2 = q^2/M^2$).

In a phantom gravitating field theory one or more of the matter fields appear in the action with an unusual sign of the kinetic term, so that they are coupled repulsively to gravity. In the case of “phantom EMD” theory, which is also a short term for the theory, we may have a number of ways the matter fields are coupled to gravity: Einstein-anti-Maxwell-anti-dilaton, Einstein-Maxwell-anti-dilaton, Einstein-Maxwell-dilaton and so on. The presence of phantom fields continues to receive support from both collected observational data [26] and theoretical models [27].

The static, spherically symmetric black hole solutions to EMD theory with phantom Maxwell and/or dilaton field were derived and their causal structure was analyzed, among which one finds nine classes of asymptotically flat and two classes of non-asymptotically flat phantom black holes [28]. In a subsequent work [29], these solutions have been generalized to multicenter solutions of phantom EMD. Recently, their thermodynamic prop-

¹This is justified since the differential equation satisfied by Weierstrass elliptic functions $\wp(\Theta)$ is just (1.2) upon replacing ρ by \wp .

erties and stability were investigated too [30]. One of the remaining tasks is to investigate their null geodesics to see how phantom fields may affect the light paths, particularly the angles of deflection, the photon spheres and related lensing effects. Deviations of the angle of deflection from the Schwarzschild value are generally due to extensions in the theory (inclusion of Maxwell fields or and scalar ones, cosmological constant and so on), departure from spherical symmetry or motion of the solution itself (mostly rotation). In this paper we examine the case due to the inclusion of (anti)-dilaton and/or (anti)-Maxwell fields.

In Section 2 we consider the cosh and sinh black hole solutions of the generalized phantom EMD, which depend on three parameters (M, q, γ) . We evaluate, and discuss, the angle of deflection $\delta\phi$ to the second order of approximation in the inverse of the impact parameter as function of the black hole three parameters. Figures, relying on exact formulas, depicting $\delta\phi$ for phantom and normal black holes are plotted against the Schwarzschild angle of deflection for different values of the parameters. The relative discrepancy is also discussed and plotted showing high values from some set of the parameters.

In Section 3 we introduce the Weierstrass elliptic functions and use and develop the method based on Weierstrass polynomial to determine exactly all kind of null geodesics to any spherically symmetric geometry provided the equation of (planar) motion of light rays may be brought to (1.2). Applications are considered in Sections 4 and 5. In the former section we consider the case $\gamma = 1$ and show that the problem of determining the null geodesics of normal Reissner-Nordström black holes by the method based on Weierstrass polynomial, which was initiated in [19], is tractable analytically and extend the analysis to phantom Reissner-Nordström black holes upon applying the results of Section 3, and in the latter one we consider the case $\gamma = 0$ and determine all the null geodesics of the phantom cosh and normal sinh EMD black holes by mere comparison to the work done in Section 3. In Sections 3 to 5, we do not aim to go into the details of each null geodesic motion; rather, we present a general procedure (Section 3) by which we discuss some type of null geodesic motions, the nature of existing divergencies and present exact reference and standard formulas for specific geodesics and for the angle of deflection. The paper ends with a Conclusion Section and an Appendix one.

2 The deflection angle of light paths in the cosh-sinh solutions of EMD

The action for EMD theory with phantom Maxwell and/or dilaton field reads

$$S = - \int d^4x \sqrt{-g} [\mathcal{R} - 2\eta_1 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \eta_2 e^{2\lambda\phi} F_{\mu\nu} F^{\mu\nu}], \quad (2.1)$$

where λ is the real dilaton-Maxwell coupling constant, and $\eta_1 = \pm 1$, $\eta_2 = \pm 1$. Normal EMD corresponds to $\eta_2 = \eta_1 = +1$, while phantom couplings of the dilaton field ϕ or/and Maxwell field $F = dA$ are obtained for $\eta_1 = -1$ or/and $\eta_2 = -1$.

The metrics of the so called cosh and sinh solutions, derived in [28], take the form

$$ds^2 = f_+ f_-^\gamma dt^2 - f_+^{-1} f_-^{-\gamma} dr^2 - r^2 f_-^{1-\gamma} d\Omega^2 \quad (2.2)$$

$$F = -\frac{q}{r^2} dr \wedge dt, \quad e^{-2\lambda\phi} = f_-^{1-\gamma}, \quad f_\pm = 1 - \frac{r_\pm}{r}, \quad \gamma = \frac{1 - \eta_1 \lambda^2}{1 + \eta_1 \lambda^2} \quad (2.3)$$

$$\eta_2(1 + \eta_1 \lambda^2) < 0 \text{ for cosh, } \eta_2(1 + \eta_1 \lambda^2) > 0 \text{ for sinh} \quad (2.3)$$

$$\gamma \in (-\infty, -1) \cup [1, +\infty) \text{ if } \eta_1 = -1 \quad (2.4)$$

$$\gamma \in (-1, +1] \text{ if } \eta_1 = +1 \quad (2.5)$$

where we have introduced the parameter γ following the notation of [30]².

These are asymptotically flat spherically symmetric black holes of mass M , electric charge q and event horizon $r_+ > 0$ related by [28]

$$M = \frac{r_+ + \gamma r_-}{2} \quad (2.6)$$

$$q = \pm \sqrt{\frac{1+\gamma}{2}} \eta_2 r_+ r_- \quad (2.7)$$

where we have substituted, into the original formula of q , $1 + \eta_1 \lambda^2 = 2/(1 + \gamma)$. Since q is real, r_- and $\eta_2(1 + \gamma)$ must have the same sign. Using this fact in (2.3), we have $r_- < 0$ for the cosh solution and $r_- > 0$ for the sinh one.

As shown in subsection 4.1 case 2. (d) of [28], $r = 0$ corresponds to a singularity for the cosh solution where geodesics terminate (a Penrose diagram is given in figure 1 of [28]). Similarly, in subsection 4.3 case 1. (a) (ii) of [28] it is established that, for generic values of $1 + \eta_1 \lambda^2$ as this is the case for $\gamma = 0$ (to which we restrict ourselves in Section 5), $r = r_-$ is a null singularity for the sinh solution (a Penrose diagram is given in figure 3 of [28]). The curvature scalar of (2.2) diverges at these two points for $\gamma = 0$

$$\mathcal{R} = -\frac{r_-(r - r_+)}{2r^3(r - r_-)^2}.$$

Expressing (r_+, r_-) in terms of M and $a^2 = q^2/M^2$, one obtains

$$r_+ = 2M, \quad r_- = \eta_2 M a^2, \quad \text{if } \gamma = 0 \quad (2.8)$$

$$r_+ = M + \mathcal{M}, \quad r_- = \frac{M - \mathcal{M}}{\gamma} = \frac{2\eta_2 M^2 a^2}{(1 + \gamma)r_+}, \quad \mathcal{M} = M \sqrt{1 - \frac{2\eta_2 \gamma a^2}{(1 + \gamma)}}, \quad \forall \gamma > -1. \quad (2.9)$$

[The limit $\gamma \rightarrow 0$ in (2.9) leads to (2.8)].

²For the sinh solution the case $\eta_2(1 + \eta_1 \lambda^2) < 0$, which would lead to $r_- < 0$, is not possible [28].

Related to the two Killing vectors $(\partial_t, \partial_\phi)$ are the two constants of motion (E, L) given by

$$f_+ f_-^\gamma \frac{dt}{d\tau} = E \quad (2.10)$$

$$r^2 \sin^2 \theta f_-^{1-\gamma} \frac{d\phi}{d\tau} = L. \quad (2.11)$$

Since (2.2) is endowed with spherical symmetry, the motion happens in a plane through the origin. Letting the plane be $\theta = \pi/2$ and inserting (2.10, 2.11) into the line element (2.2):

$$f_+ f_-^\gamma \left(\frac{dt}{d\tau} \right)^2 - f_+^{-1} f_-^{-\gamma} \left(\frac{dr}{d\tau} \right)^2 - r^2 f_-^{1-\gamma} \left(\frac{d\phi}{d\tau} \right)^2 = \varepsilon$$

(with $\varepsilon = 1, 0$ for massive, massless particles, respectively) we bring it to

$$\left(\frac{dr}{d\tau} \right)^2 = E^2 - f_+ f_-^\gamma \left[\varepsilon + \frac{L^2}{r^2 f_-^{1-\gamma}} \right]. \quad (2.12)$$

For scattering states $E^2 - \varepsilon > 0$. Eliminating τ in (2.11, 2.12) we arrive at

$$\left(\frac{dr}{d\phi} \right)^2 = \frac{r^4 f_-^{2(1-\gamma)}}{L^2} \left[E^2 - f_+ f_-^\gamma \left[\varepsilon + \frac{L^2}{r^2 f_-^{1-\gamma}} \right] \right]$$

which we re-write in terms of $u = 1/r$ as

$$\left(\frac{du}{d\phi} \right)^2 = \frac{f_-^{2(1-\gamma)}}{L^2} \left[E^2 - f_+ f_-^\gamma \left[\varepsilon + \frac{L^2 u^2}{f_-^{1-\gamma}} \right] \right]. \quad (2.13)$$

From now on we take $\varepsilon = 0$ so that the condition for light scattering is $E^2 > 0$. Now, let $g(u)$ be the function

$$g(u) := L^2 u^2 f_+ f_-^{2\gamma-1}$$

and $u_n = 1/r_n$ be the point on the light scattering geodesic nearest the origin where $\frac{du}{d\phi}(u_n) = 0$. Since $E^2 = g(u_n)$, the angle of deflection, which is twice the variation of ϕ minus π , takes the form

$$\delta\phi = 2L \int_0^{u_n} \frac{du}{f_-^{1-\gamma}(u) \sqrt{g(u_n) - g(u)}} - \pi. \quad (2.14)$$

Setting $u = u_n x$, we bring (2.14) to the form

$$\delta\phi = 2 \int_0^1 \frac{L u_n \sqrt{1-x^2}}{f_-^{1-\gamma}(u) \sqrt{g(u_n) - g(u)}} \frac{dx}{\sqrt{1-x^2}} - \pi. \quad (2.15)$$

With $u_n \ll 1$ for scattering with large values of the impact parameter ($b = L/E$), we obtain

$$\begin{aligned} \frac{Lu_n \sqrt{1-x^2}}{f_-^{1-\gamma}(u) \sqrt{g(u_n) - g(u)}} &= 1 + \left[\frac{[(\gamma-1)r_- + 2M]}{2} \frac{1-x^3}{1-x^2} - (\gamma-1)r_- x \right] u_n \\ &+ \frac{1}{8(1+x)^2} \left\{ 3r_+^2(1+x+x^2)^2 + 2r_- r_+ [2\gamma-1+2\gamma x + (6\gamma-1)x^2 + 2x^3 + x^4] \right. \\ &\quad \left. + r_-^2 [4\gamma^2 - 1 + 6(2\gamma-1)x + (8\gamma^2+1)x^2 + 6x^3 + 3x^4] \right\} u_n^2 + O[u_n]^3 \end{aligned}$$

where we have used (2.6): $r_+ + \gamma r_- = 2M$. Performing the integrations over x we obtain

$$\delta\phi = \frac{4M}{r_n} + \left\{ -2M[r_+ + r_-(2\gamma-1)] + \frac{\pi}{16} [15r_+^2 + 6r_- r_+ (4\gamma-1) + r_-^2 (16\gamma^2-1)] \right\} \frac{1}{r_n^2} + O[1/r_n]^3. \quad (2.16)$$

Since the values of γ depend on η_1 according to (2.4, 2.5) and the sign of r_- is that of $\eta_2(1+\gamma)$ by (2.7), the deflection angle depends on the type of EMD under investigation. From (2.8, 2.9) one sees that, for both cases $\gamma = 0$ and $\gamma \neq 0$, the limit case $q = 0$ corresponds to $r_- = 0$ and $r_+ = 2M$. Thus in the limit $q \rightarrow 0$, $\delta\phi$ approaches the value $\delta\phi(r_- = 0, r_+ = 2M)$, which is the Schwarzschild angle of deflection $\delta\phi_S$:

$$\lim_{q \rightarrow 0} \delta\phi = \delta\phi(r_- = 0, r_+ = 2M) = \delta\phi_S = \frac{4M}{r_n} + \frac{(15\pi-16)M^2}{4} \frac{1}{r_n^2} + O[1/r_n]^3. \quad (2.17)$$

Using this along with (2.8, 2.9) in (2.16) we express $\delta\phi$ in terms of the charges (M, q) and $\delta\phi_S$

$$\delta\phi = \delta\phi_S - \frac{\pi M^2 a^2}{16} \left[\eta_2 \frac{4(3\pi-8)}{\pi} + a^2 \right] \frac{1}{r_n^2} + \dots, \quad \text{if } \gamma = 0 \quad (2.18)$$

$$\delta\phi = \delta\phi_S - \eta_2 \left[\frac{(\gamma-1)[16\gamma - \pi(\gamma+1)]M|M-\mathcal{M}| + \pi\gamma(7\gamma-1)q^2}{8\gamma^2} \right] \frac{1}{r_n^2} + \dots, \quad \forall \gamma > -1 \quad (2.19)$$

$$\begin{aligned} &= \frac{4M}{r_n} + \left\{ -2M(M+\mathcal{M}) + \eta_2 \frac{4(1-2\gamma)a^2 M^3}{(1+\gamma)(M+\mathcal{M})} \right. \\ &\quad \left. + \frac{\pi}{16} \left[15(M+\mathcal{M})^2 + \frac{4(16\gamma^2-1)a^4 M^4}{(1+\gamma)^2(M+\mathcal{M})^2} + \eta_2 \frac{12(4\gamma-1)a^2 M^2}{1+\gamma} \right] \right\} \frac{1}{r_n^2} + \dots, \quad \forall \gamma > -1 \end{aligned} \quad (2.20)$$

where we have made use of $M - \mathcal{M} = \eta_2 |M - \mathcal{M}|$ and (2.9). [The limit $\gamma \rightarrow 0$ in (2.19) or in (2.20) leads to (2.18)]. Thus to the first order of approximation in $1/r_n$ all normal and phantom black holes deflect light paths in the same way with $\delta\phi = 4M/r_n + \dots$. To the second order of approximation in $1/r_n$, the added contribution to the Schwarzschild one (second terms in (2.18, 2.19)) does not depend on the sign of q but depends on the signs of (η_1, η_2) .

First consider the special case $\gamma = 0$, which corresponds to $\eta_1 = +1$. In normal EMD ($\eta_2 = +1$) we have $\delta\phi < \delta\phi_S$. In phantom EMD ($\eta_2 = -1$), which is E-anti-MD theory, we have $\delta\phi > \delta\phi_S$ provided we restrict ourselves to the physical case $a^2 < 1$ ($4(3\pi-8)/\pi \simeq 1.8$). Thus, in the presence of phantom fields, light rays are more deflected than in the normal case. Phantom fields causes light rays to bent with an angle $(3\pi-8)q^2/(2r_n^2)$ larger than the angle of deflection caused by normal fields. This difference is independent on the sign of q but

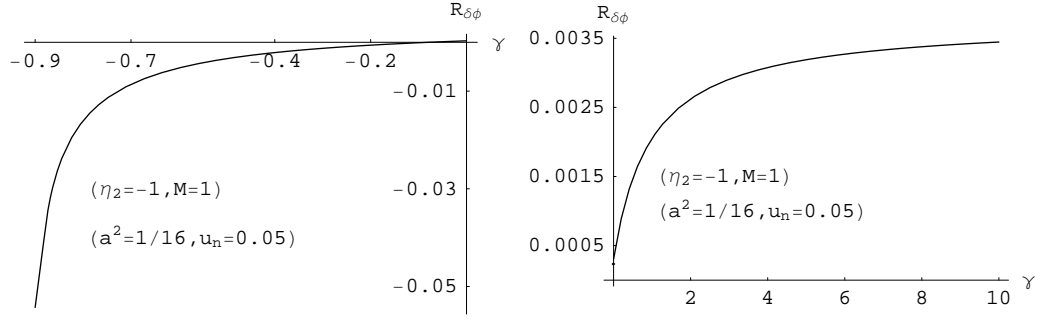


Figure 1: The relative discrepancy $R_{\delta\phi} = (\delta\phi - \delta\phi_S)/\delta\phi_S$ vs. γ for fixed $(M = 1, a^2 = 1/16, u_n = 0.05)$ for E-anti-M-(anti)-D. $R_{\delta\phi}$ is not always negative for $\gamma < 0$: it vanishes then becomes positive for some γ_0 between -0.1 and -0.05 .

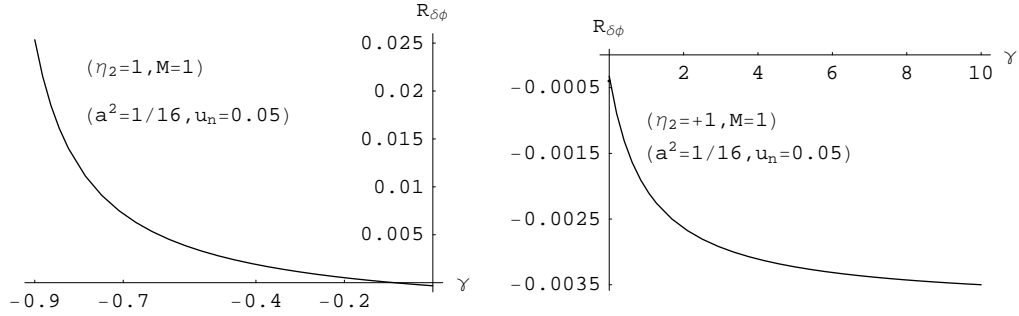


Figure 2: The relative discrepancy $R_{\delta\phi} = (\delta\phi - \delta\phi_S)/\delta\phi_S$ vs. γ for fixed $(M = 1, a^2 = 1/16, u_n = 0.05)$ for EM-(anti)-D. $R_{\delta\phi}$ is not always positive for $\gamma < 0$: it vanishes then becomes negative for some γ_0 between -0.1 and -0.05 .

depends on the mass of the black hole through r_n . Using (2.17) we obtain

$$\frac{3\pi - 8}{2r_n^2} M^2 a^2 \simeq \frac{3\pi - 8}{32} a^2 (\delta\phi_S)^2 \quad (2.21)$$

which is 0.22% of the Schwarzschild value $\delta\phi_S$ if $M = 1, a^2 = 1/4, u_n = 0.05$ and 0.89% of it (the exact value is 1.03) if $M = 1, a^2 = 1, u_n = 0.05$.

The case $\gamma = 1$ is phantom or normal Reissner-Nordström black hole. With

$$\delta\phi = \delta\phi_S - \eta_2 \frac{3\pi M^2 a^2}{4} \frac{1}{r_n^2} + \dots, \quad \text{if } \gamma = 1 \quad (2.22)$$

we confirm the previous conclusions: $\delta\phi < \delta\phi_S$ for normal Reissner-Nordström black holes and $\delta\phi > \delta\phi_S$ for phantom ones. A phantom Reissner-Nordström black hole deflects light with an angle $3\pi q^2/(2r_n^2)$ larger than the deflection angle caused by a normal Reissner-Nordström black hole

$$\frac{3\pi}{2r_n^2} M^2 a^2 \simeq \frac{3\pi}{32} a^2 (\delta\phi_S)^2 \quad (2.23)$$

which is 1.47 % of the Schwarzschild value $\delta\phi_S$ if $M = 1, a^2 = 1/4, u_n = 0.05$ and 5.89 % of it (the exact value is 6.56) if $M = 1, a^2 = 1, u_n = 0.05$.

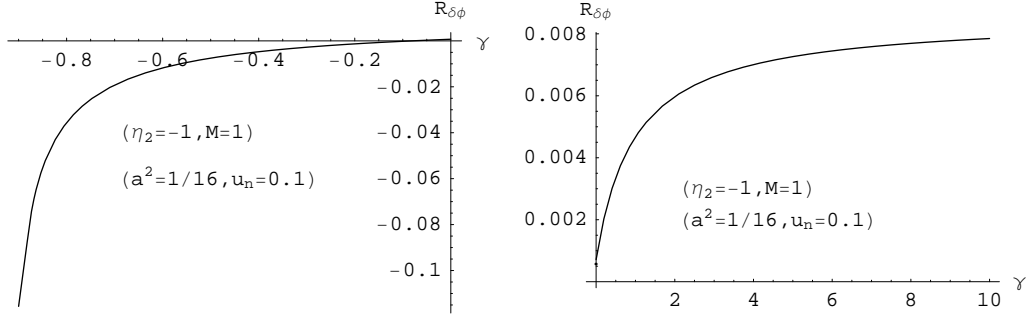


Figure 3: The relative discrepancy $R_{\delta\phi} = (\delta\phi - \delta\phi_S)/\delta\phi_S$ vs. γ for fixed $(M = 1, a^2 = 1/16, u_n = 0.1)$ for E-anti-M-(anti)-D. $R_{\delta\phi}$ is not always negative for $\gamma < 0$: it vanishes then becomes positive for some γ_0 between -0.1 and -0.05 .

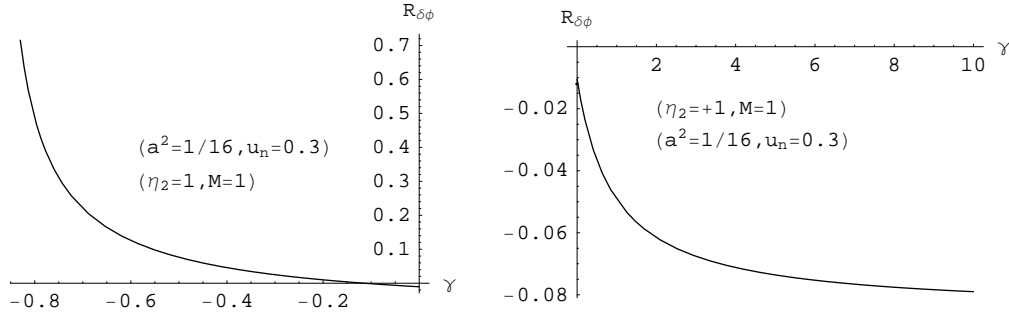


Figure 4: The relative discrepancy $R_{\delta\phi} = (\delta\phi - \delta\phi_S)/\delta\phi_S$ vs. γ for fixed $(M = 1, a^2 = 1/16, u_n = 0.3)$ for EM-(anti)-D. $R_{\delta\phi}$ is not always positive for $\gamma < 0$: it vanishes then becomes negative for some γ_0 between -0.15 and -0.1 .

Now, consider the case $\gamma \neq 0$ and $\gamma \neq 1$. Here again we confirm the previous conclusions: $\delta\phi < \delta\phi_S$ for normal black holes and $\delta\phi > \delta\phi_S$ for phantom ones provided γ is large enough. This is no longer true if γ is closer to -1 as the coefficient of $1/r_n^2$ becomes too large invalidating (2.20). Let us look at the derivative of the relative discrepancy function $R_{\delta\phi}$

$$R_{\delta\phi} = \frac{\delta\phi - \delta\phi_S}{\delta\phi_S}, \quad (2.24)$$

$$\frac{dR_{\delta\phi}}{d\gamma} = \frac{2LE^2}{\delta\phi_S} \int_0^{u_n} \frac{\ln(f_-) du}{f_-^{1-\gamma}(u) [E^2 - g(u)]^{3/2}}, \quad (2.25)$$

which is positive for phantom black holes ($f_- > 1$) and negative for normal ones ($f_- < 1$). Thus, $R_{\delta\phi}(\gamma)$ (all other parameters being fixed) increases for phantom black holes and decreases for normal ones. Figures (1 to 4)), which have been plotted using the exact formula (2.15), show the existence of a zero γ_0 beyond which $R_{\delta\phi} > 0$ ($\delta\phi > \delta\phi_S$) for phantom black holes and $R_{\delta\phi} < 0$ ($\delta\phi < \delta\phi_S$) for normal ones.

Figures (1 to 4) have been plotted for fixed $(M = 1, a^2 = 1/16)$ and relatively large values of r_n ($u_n = 0.05, 0.1$) or small values of r_n up to the photon sphere ($u_n = 0.3$). Based on these and on some other figures (not shown in this paper) for larger values of a^2 up to 1 and different values of u_n , we can draw a general conclusion: All the other parameters being fixed, there is always a root γ_0 in the interval $(-0.2, 0)$ to $R_{\delta\phi}(\gamma) = 0$. Said otherwise, for some values of γ in the interval $(-0.2, 0)$, it seems there is always a critical value $r_n = r_c$, larger than the photon sphere, where $R_{\delta\phi} = 0$.

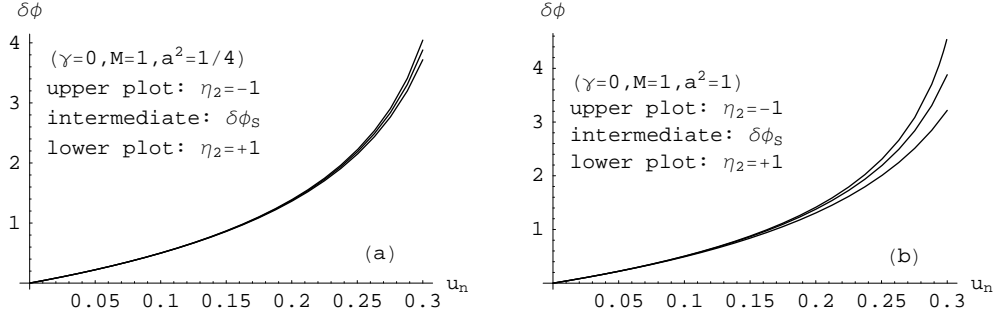


Figure 5: The angle of deflection (Eq. (2.15)) in radians vs. $u_n = 1/r_n$ (r_n is the point on the null geodesic nearest the origin). In both plots the intermediate graph is the Schwarzschild value $\delta\phi_S$. (a): Phantom EMD cosh ($\eta_2 = -1$) and normal EMD sinh ($\eta_2 = +1$) black holes for $\gamma = 0 > \gamma_0$, $M = 1$ and $a^2 = 1/4$. (b): Phantom EMD cosh ($\eta_2 = -1$) and normal EMD sinh ($\eta_2 = +1$) black holes for $\gamma = 0$, $M = 1$ and $a^2 = 1$.

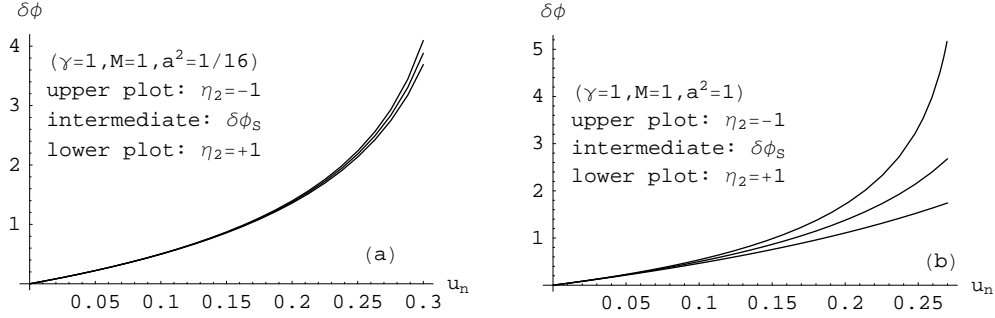


Figure 6: The angle of deflection (Eq. (2.15)) in radians vs. $u_n = 1/r_n$ (r_n is the point on the null geodesic nearest the origin). In both plots the intermediate graph is the Schwarzschild value $\delta\phi_S$. (a): Phantom Reissner-Nordström ($\eta_2 = -1$) and normal Reissner-Nordström ($\eta_2 = +1$) black holes for $\gamma = 1 > \gamma_0$, $M = 1$ and $a^2 = 1/16$. (b): Phantom Reissner-Nordström ($\eta_2 = -1$) and normal Reissner-Nordström ($\eta_2 = +1$) black holes for $\gamma = 1$, $M = 1$ and $a^2 = 1$.

As $\gamma \rightarrow +\infty$, $R_{\delta\phi}$ approaches the limit

$$\frac{2}{\delta\phi_S} \int_0^1 \frac{e^{K_-x} dx}{\sqrt{e^{K_-}(1-K_+) - x^2 e^{K_-x}(1-K_+x)}} - \frac{\pi + \delta\phi_S}{\delta\phi_S}$$

$$K_{\pm} = Mu_n(\sqrt{1 - 2\eta_2 a^2} \pm 1).$$

As Figures (5 to 8) reveal, the vertical spacing $|\delta\phi - \delta\phi_S|$ depends slightly on γ , which itself depends on η_1 , and increases with a^2 . In the extreme case ($a^2 = 1$), the winding number for phantom black holes with $\eta_2 = -1$ and $\gamma > \gamma_0$ (regardless the sign of η_1) diverges near $u_n \simeq 0.3$, a value for which the angle of deflection for phantom ($\eta_1 = -1$) or normal ($\eta_1 = +1$) black holes with $\eta_2 = +1$ is less than few radians. As we shall see later, this is a consequence of the fact that the photon sphere for black holes with $\eta_2 = -1$ (black holes where the Maxwell field F is coupled repulsively to gravity) and $\gamma > \gamma_0$ is larger than $3M$, which is the Schwarzschild limit, allowing photons to orbit the hole at larger, ever-decreasing, radii. The Schwarzschild limit $3M$ is larger than the photon sphere for black holes with $\eta_2 = +1$ and $\gamma > \gamma_0$. If $\gamma < \gamma_0$, all that said for black holes with $\eta_2 = -1$ (respectively $\eta_2 = +1$) applies to black holes with $\eta_2 = +1$ (respectively $\eta_2 = -1$).

It is useful to express $\delta\phi$ in terms of the charges (M, q) and the impact parameter $b = L/E$. Solving the

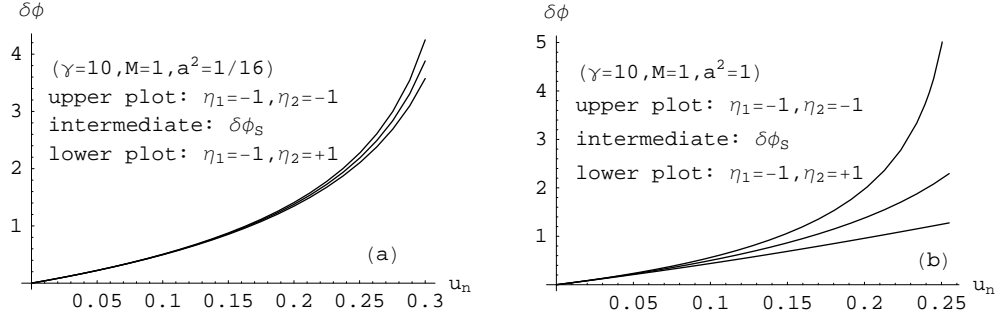


Figure 7: The angle of deflection (Eq. (2.15)) in radians vs. $u_n = 1/r_n$ (r_n is the point on the null geodesic nearest the origin). In both plots the intermediate graph is the Schwarzschild value $\delta\phi_S$. (a): E-anti-M-anti-D and EM-anti-D black holes for $\gamma = 10 > \gamma_0$, $M = 1$ and $a^2 = 1/16$. (b): E-anti-M-anti-D and EM-anti-D black holes for $\gamma = 10$, $M = 1$ and $a^2 = 1$.

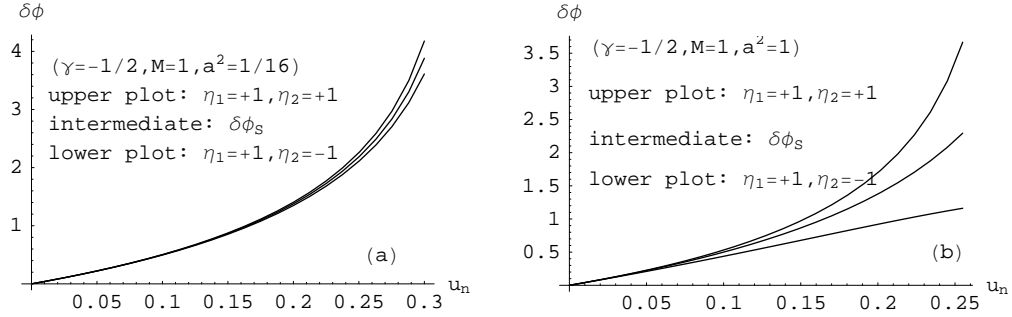


Figure 8: The angle of deflection (Eq. (2.15)) in radians vs. $u_n = 1/r_n$ (r_n is the point on the null geodesic nearest the origin). In both plots the intermediate graph is the Schwarzschild value $\delta\phi_S$. (a): E-anti-MD and EMD black holes for $\gamma = -1/2 < \gamma_0$, $M = 1$ and $a^2 = 1/16$. (b): E-anti-MD and EMD black holes for $\gamma = -1/2$, $M = 1$ and $a^2 = 1$.

equation $E^2 = g(u_n)$ for $u_n = 1/r_n$ to the second order, we obtain

$$u_n = \frac{1}{b} + \frac{r_+ + r_- (2\gamma - 1)}{2} \frac{1}{b^2} + O[1/b]^3.$$

Using this in (2.16) we derive the desired equation

$$\delta\phi = \frac{4M}{b} + \frac{\pi}{16} [15r_+^2 + 6r_- r_+ (4\gamma - 1) + r_-^2 (16\gamma^2 - 1)] \frac{1}{b^2} + O[1/b]^3. \quad (2.26)$$

The Schwarzschild value, corresponding to the limit $q \rightarrow 0$, is again given by

$$\delta\phi_S = \delta\phi(r_- = 0, r_+ = 2M) = \frac{4M}{b} + \frac{15\pi M^2}{4} \frac{1}{b^2} + O[1/b]^3. \quad (2.27)$$

In terms of $\delta\phi_S$, as given in (2.27), b and (M, q) , the expressions (2.18, 2.19, 2.22) become, respectively

$$\delta\phi = \delta\phi_S - \frac{\pi q^2}{16} \left[12\eta_2 + \frac{q^2}{M^2} \right] \frac{1}{b^2} + \dots, \quad \text{if } \gamma = 0 \quad (2.28)$$

$$\delta\phi = \delta\phi_S - \eta_2 \left[\frac{(\gamma-1)[16\gamma - \pi(\gamma+1)]M|M - \mathcal{M}| + [\pi\gamma(7\gamma-1) + 16\gamma^2]q^2}{8\gamma^2} \right] \frac{1}{b^2} + \dots, \quad \forall \gamma > -1 \quad (2.29)$$

$$\delta\phi = \delta\phi_S - \eta_2 \frac{(3\pi+8)q^2}{4} \frac{1}{b^2} + \dots, \quad \text{if } \gamma = 1. \quad (2.30)$$

3 Determination of geodesics by Weierstrass elliptic functions: general procedure

Any differential equation of the form (1.2) has a unique solution in terms of the Weierstrass elliptic functions of the form [2, 3]

$$\rho = \wp(\Theta + C)$$

where C is generally a complex constant.

$\wp(\Theta)$ is an even single-valued doubly-periodic function with half periods (ω, ω') chosen in such a way that $\text{Im}(\omega'/\omega) > 0$. When the Weierstrass polynomial $w(\rho) = 4\rho^3 - g_2\rho - g_3 = 4(\rho - e_1)(\rho - e_2)(\rho - e_3)$ has three real roots (e_3, e_2, e_1) , there are three half periods $(\omega_1, \omega_2, \omega_3)$ depending on (ω, ω') such that

$$\wp(\omega_k) = e_k, \quad (k = 1, 2, 3). \quad (3.1)$$

In order to have $e_3 < e_2 < e_1$ we choose the three half periods $(\omega_1, \omega_2, \omega_3)$ to satisfy [2]

$$\omega_1 \equiv \omega > 0, \quad \omega_3 \equiv \omega' \text{ (with } -i\omega' > 0), \quad \omega_2 \equiv -\omega - \omega' = -\omega_1 - \omega_3. \quad (3.2)$$

The expression of ω_2 is a consequence of $e_1 + e_2 + e_3 = 0$.

3.1 Three distinct real roots

The Weierstrass polynomial $w(\rho)$ will have three real roots if

$$g_2 > 0 \text{ and } \Delta \equiv g_2^3 - 27g_3^2 > 0. \quad (3.3)$$

We parameterize the (real) roots by the angle $0 \leq \eta \leq \pi$ as follows

$$e_3 = -\sqrt{\frac{g_2}{3}} \cos\left(\frac{\pi - \eta}{3}\right) < 0, \quad e_2 = -\sqrt{\frac{g_2}{3}} \cos\left(\frac{\pi + \eta}{3}\right), \quad e_1 = \sqrt{\frac{g_2}{3}} \cos\left(\frac{\eta}{3}\right) > 0 \quad (3.4)$$

$$\cos \eta = \frac{9g_3}{\sqrt{3g_2^3}}, \quad \sin \eta = \sqrt{\frac{\Delta}{g_2^3}} > 0.$$

The signs of $e_3 < 0$, $e_1 > 0$ and $\sin \eta > 0$ are well defined, $e_3 < e_2 < e_1$ and the sign of e_2 depends on g_3 :

$$e_2 \cdot g_3 < 0 \text{ and } e_2 = 0 \text{ if } g_3 = 0 \quad (g_3 = 4e_1e_2e_3). \quad (3.5)$$

Motion is possible where $w(\rho) \geq 0$:

$$e_3 \leq \rho \leq e_2 \text{ or } \rho \geq e_1. \quad (3.6)$$

Conversely, we can reverse (3.1) and express the half periods $(\omega_1, \omega_2, \omega_3)$ in terms of the roots³ [2].

$$\omega_1 = \int_{e_1}^{\infty} \frac{d\rho}{\sqrt{w(\rho)}} = \int_{e_3}^{e_2} \frac{d\rho}{\sqrt{w(\rho)}} \quad (3.7)$$

$$\omega_3 = \int_{e_3}^{\infty} \frac{d\rho}{\sqrt{w(\rho)}} = i \int_{e_2}^{e_1} \frac{d\rho}{\sqrt{-w(\rho)}}. \quad (3.8)$$

Let ρ_{∞} , ρ_0 be the values of ρ corresponding to $r = +\infty$, $r = 0$, respectively, and let ρ_+ , ρ_- correspond to $r = r_+$, $r = r_-$, if there are any⁴. A singularity is denoted by ρ_{sing} (r_{sing}), which may be any of ρ_0 , ρ_- depending on the theory. In a general physical situation $(\rho_{\infty}, \rho_0, \rho_+, \rho_-, \dots, e_3, e_2, e_1)$ are functions⁵ of the vector of parameters $\vec{p} = (\text{charges, constants of motion}) = (M, q, \dots, E, L, \dots)$ so that the locations of these points on the ρ -axis change with \vec{p} . We shall represent (e_3, e_2, e_1) at the same locations on the ρ -axis while $(\rho_{\infty}, \rho_0, \rho_+, \rho_-, \dots)$ appear on different locations depending on \vec{p} .

In order to determine all types of geodesic motion in a given geometry, one has to consider all allowed possible locations of $(\rho_{\infty}, \rho_0, \rho_+, \rho_-, \dots)$ with respect to (e_3, e_2, e_1) . Once this is done, any geodesic motion that can be brought to (1.2) is integrated by mere comparison with the work done in this section. We shall provide some examples in this section and in the next two we apply the procedure to EMD and to Reissner-Nordström phantom and normal black holes. In order to illustrate the procedure, we shall envisage only some locations of $(\rho_{\infty}, \rho_0, \rho_+, \rho_-, \dots)$ with respect to (e_3, e_2, e_1) most of which are related to EMD Reissner-Nordström phantom and normal black holes.

3.1.1 Scattering and trapped paths

We consider four possible situations.

(a) $e_3 < \rho_{\infty} < e_2 < e_1 < \rho_0 = \rho_{\text{sing}}$. As light scatters from $r = +\infty \rightarrow r_n \rightarrow r = +\infty$, the corresponding point on the ρ -axis moves from $\rho_{\infty} \rightarrow e_2 \rightarrow \rho_{\infty}$, if $\rho(r)$ is a decreasing function of r , or from $\rho_{\infty} \rightarrow e_3 \rightarrow \rho_{\infty}$, if $\rho(r)$ is an increasing function of r . We consider the former case throughout this section, which is too going to be the case for the next two sections. The solution to (1.2) is

$$\rho(r) = \wp(\Theta(\phi) + C).$$

³There is a third expression for ω_3 which appears with a misprinted sign in [3, 19]. The correct expression is: $\omega_3 = +i \int_{-\infty}^{e_3} \frac{d\rho}{\sqrt{-w(\rho)}}$.

⁴In case of wormhole solutions, one introduces ρ_a corresponding to the radius a of the throat.

⁵Some of which may be constants as in the case of Schwarzschild solution where $e_3 < \rho_{\infty} = -1/12 < e_2$ and $\rho_0 = +\infty$.

To fix C we may assume $\Theta = 0$ at $\rho_n = e_2$, corresponding to r_n ; or assume $\Theta = 0$ at ρ_∞ . The former case looks simpler leading to $e_2 = \wp(C)$ and thus, by (3.1), $C = \omega_2$ or $C = -\omega_2$ (\wp is an even function!). We choose the latter solution so that by (3.2) $C = \omega_1 + \omega_3$ and

$$\rho(r) = \wp(\Theta(\phi) + \omega_1 + \omega_3). \quad (3.9)$$

The angle of deflection is

$$\delta\phi = 2 \int_{\rho_\infty}^{e_2} \frac{d\rho}{\sqrt{w(\rho)}} - \pi = 2 \left(\int_{e_3}^{e_2} - \int_{e_3}^{\rho_\infty} \right) \frac{d\rho}{\sqrt{w(\rho)}} - \pi = 2\omega_1 - 2 \int_{e_3}^{\rho_\infty} \frac{d\rho}{\sqrt{w(\rho)}} - \pi \quad (3.10)$$

where we have used the second formula in (3.7).

The other possible motion, the so called trapped or terminating bound path, is in the region between e_1 and ρ_0 where $w(\rho) \geq 0$. If the path starts from the singularity $\rho_0 = \rho_{\text{sing}} (r = 0)$ it reaches the farthest point $\rho_f = e_1$ ($r = r_f$) and then back to ρ_0 . Again choosing $\Theta = 0$ at the farthest point, the solution is

$$\rho(r) = \wp(\Theta(\phi) + C), \text{ with } C = \omega_1 \text{ (or } C = -\omega_1). \quad (3.11)$$

(b) $e_3 < \rho_\infty < e_2 < e_1 < \rho_+ < \rho_0 < \rho_- = +\infty$. If ρ_0 is a singularity, then this case is identical to (a) with a scattering path from $\rho_\infty \rightarrow e_2 \rightarrow \rho_\infty$ given by (3.9) and a trapped path between e_1 and ρ_0 given by (3.11).

If ρ_- is a singularity but ρ_0 is not, then there is a trapped path between e_1 and ρ_- given by (3.11).

If neither ρ_0 nor ρ_- is a singularity, the path is a many-world periodic bound orbit [16] in that the path, after crossing the inner horizon at $r = r_-$, emerges in another copy of the space-time then in another copy of it and so on. If we choose $\Theta = 0$ at $\rho = e_1$, then the solution will be given by (3.11).

(c) $e_2 < \rho_0 < e_1 < \rho_\infty < \rho_+ < \rho_- = +\infty$. Since $w(\rho) < 0$ for $\rho \in (e_2, e_1)$, there are no paths that can reach or emanate from the origin.

There is a path that extends from spatial infinity (ρ_∞) to the inner horizon (ρ_-). This is not a spiral path since the integral

$$\int_{\text{constant} \geq \rho_\infty}^{\infty} d\rho / \sqrt{w(\rho)}$$

converges (Θ remains finite). If ρ_- is not a singularity, the path is called a two-world scattering orbit in that the path emerges, after crossing the inner horizon at $r = r_-$, in another copy of the space-time and back to spatial infinity. If ρ_- is a singularity, we have an absorbed path from spatial infinity to the singularity. The solution is again $\rho(r) = \wp(\Theta(\phi) + C)$. Since there is no farthest or nearest point on the path, we choose $\Theta = 0$ at ρ_∞ leading to $\rho_\infty = \wp(C)$. Using the inverse function to \wp , $C = \int_{\rho_\infty}^{\infty} d\rho / \sqrt{w(\rho)}$ and

$$\rho(r) = \wp(\Theta(\phi) + C), \text{ with } C = \int_{\rho_\infty}^{\infty} d\rho / \sqrt{w(\rho)}. \quad (3.12)$$

(d) $\rho_0 < e_3 < \rho_\infty < e_2 < e_1 < \rho_- = +\infty$. Here again no paths that can reach or emanate from the origin.

We have a scattering path from $\rho_\infty \rightarrow e_2 \rightarrow \rho_\infty$ and the solution is again given by (3.9) but with different \vec{p} .

There is another path from $r = r_1$ ($\rho = e_1$) to $r = r_-$ ($\rho = \rho_-$). Since in this case r_1 is finite ($\rho_\infty < e_1 \Rightarrow r_1 < +\infty$), the path is a many-world periodic bound orbit if ρ_- is not a singularity or a trapped path if ρ_- is a singularity. If we choose $\Theta = 0$ at $\rho = e_1$, then the solution will be given by (3.11).

3.1.2 Absorbed and circular paths

Absorbed paths extend from spatial infinity (ρ_∞) to the (nearest) singularity (ρ_{sing}). Such paths exist if both points ρ_∞ , ρ_{sing} are in $[e_3, e_2]$ or in $[e_1, +\infty)$. [There are no such paths in the Schwarzschild case when $w(\rho)$ has three distinct real roots.] It is clear that there are no circular paths when $w(\rho)$ has three distinct real roots.

3.2 Two distinct real roots

The Weierstrass polynomial $w(\rho)$ will have two real roots if

$$g_2 > 0 \text{ and } \Delta \equiv g_2^3 - 27g_3^2 = 0. \quad (3.13)$$

This happens when one of the local extreme values of $w(\rho)$ is zero. The second condition in (3.13) splits into two cases.

3.2.1 Stable circular and bound paths: $g_3 = (g_2/3)\sqrt{g_2/3} > 0$

The local maximum value of $w(\rho)$, which is at $\rho_{\text{max}} = -(1/2)\sqrt{g_2/3}$, is zero. We have

$$e_3 = e_2 = -\frac{1}{2}\sqrt{\frac{g_2}{3}}, \quad e_1 = \sqrt{\frac{g_2}{3}} \quad (\eta = 0 \text{ in (3.4)}). \quad (3.14)$$

Since at $\rho_{\text{max}} = -(1/2)\sqrt{g_2/3}$, $w(\rho)$ has a local maximum, the polynomial $P(r)$ in (1.1) has a local maximum too at the corresponding point r_{max} . But since $P(r) \propto E^2 - V(r)$ (compare with (2.12, 2.13)), the potential $V(r)$ has there a local minimum. Thus

$$\rho \equiv \rho_{\text{max}} = e_3 = -(1/2)\sqrt{g_2/3} \quad (3.15)$$

is a stable circular path.

Paths in the region $\rho \geq e_1$ are periodic: They include the periodic bound and the so called terminating bound (trapped) and many-world periodic bound orbits [16]. No matter the location of ρ_∞ with respect to e_1 is, the equation of motion can be integrated directly. Let $t = \sqrt{\rho - e_1}$ and

$$k^2 = e_1 - e_3 = (3/2)\sqrt{g_2/3} > 0 \quad (3.16)$$

then (1.2) reads

$$d\Theta = \frac{d\rho}{2(\rho - e_3)\sqrt{\rho - e_1}} = \frac{dt}{t^2 + k^2}$$

leading to

$$\sqrt{\rho - e_1} = k \tan[k(\Theta - C)] \quad \text{or} \quad \rho - e_3 = \frac{2(e_1 - e_3)}{1 + \cos[2k(\Theta - C)]}. \quad (3.17)$$

For the Schwarzschild black hole $\rho = M/(2r) - 1/12$ [19], $\Theta = \phi$, $g_2 = 1/12$, $g_3 = 1/216$, $e_3 = e_2 = -1/12$, $e_1 = 1/6$, $k = 1/2$ and the second formula in (3.17) is just Eq. (10) of [19].

For the phantom ($\eta_2 = -1$) and normal ($\eta_2 = +1$) Reissner-Nordström black holes we derive in Subsection 4.3 from the first formula in (3.17) the following orbit

$$\tan\left[\frac{\phi - C}{2}\right] = \sqrt{\frac{r_+ - r}{r - r_-}} \quad (3.18)$$

which is a trapped path for the phantom black hole ($r_+ > r > 0$) and a many-world periodic bound path for the normal black hole ($r_+ > r > r_-$). Using the double-angle formula for tan we re-write it as

$$\tan(\phi - C) = \frac{M - r}{\sqrt{2Mr - \eta_2 q^2 - r^2}}, \quad (3.19)$$

which is the correct form of the misprinted Eq. (64) of [19].

3.2.2 Unstable circular and spiral paths: $g_3 = -(g_2/3)\sqrt{g_2/3} < 0$

The local minimum value of $w(\rho)$, which is at $\rho_{\min} = +(1/2)\sqrt{g_2/3}$, is zero. We have

$$e_2 = e_1 = +\frac{1}{2}\sqrt{\frac{g_2}{3}}, \quad e_3 = -\sqrt{\frac{g_2}{3}} \quad (\eta = \pi \text{ in (3.4)}). \quad (3.20)$$

Since at $\rho_{\min} = +(1/2)\sqrt{g_2/3}$, $w(\rho)$ has a local minimum, the potential $V(r)$ has there a local maximum (compare with (2.12, 2.13)). Thus

$$\rho \equiv \rho_{\min} = e_1 = +(1/2)\sqrt{g_2/3} \quad (3.21)$$

is an unstable circular path.

Paths in the regions $e_3 \leq \rho \leq e_2$ and $\rho \geq e_1$ depend on the location of ρ_∞ . Here we consider the case $e_3 < \rho_\infty < e_2$. There are two spiral paths approaching the circle $\rho = e_1 = +(1/2)\sqrt{g_2/3}$ from 1) ρ_∞ or from 2) ρ_0 ($\rho_0 > e_1$ in this case!). The paths end orbiting the center at an ever 1) decreasing or 2) increasing radii r without, however, reaching the unstable circular path at $r = r_1$ corresponding to $\rho = e_1$. The equation of motion can be integrated directly. Let $s = \sqrt{\rho - e_3}$ and $k^2 = e_1 - e_3 = (3/2)\sqrt{g_2/3} > 0$ as in (3.16). Then (1.2) reads

$$d\Theta = \frac{d\rho}{2|\rho - e_1|\sqrt{\rho - e_3}} = \frac{ds}{|s^2 - k^2|}$$

and Θ , as well as the angle of deflection, diverge as $\ln|\rho - e_1|$ as $\rho \rightarrow e_1$, which is a general behavior in the strong field limit valid for all spherically symmetric solutions [31, 32]. Integration leads to

$$1) \sqrt{\rho - e_3} = -k \coth[k(\Theta - C)] \quad \text{and} \quad 2) \sqrt{\rho - e_3} = -k \tanh[k(\Theta - C)] \quad (3.22)$$

or

$$\rho - e_1 = -\frac{2(e_1 - e_3)}{1 \mp \cosh[2k(\Theta - C)]}, \quad (- \rightarrow 1), (+ \rightarrow 2)). \quad (3.23)$$

For the Schwarzschild black hole $\rho = M/(2r) - 1/12$ [19], $\Theta = \phi$, $g_2 = 1/12$, $g_3 = -1/216$, $e_3 = -1/6$, $e_1 = e_2 = 1/12$ and $k = 1/2$ we obtain the solutions (11) of [19].

3.3 One real root

The Weierstrass polynomial $w(\rho)$ will have one real root with multiplicity 1 if

$$\Delta \equiv g_2^3 - 27g_3^2 < 0. \quad (3.24)$$

The sign of the real root e_r

$$e_r = \frac{1}{2 \cdot 9^{1/3}} [(9g_3 + \sqrt{3}\sqrt{-\Delta})^{1/3} + (9g_3 - \sqrt{3}\sqrt{-\Delta})^{1/3}] \quad (3.25)$$

is related to that of g_3 by

$$e_r \cdot g_3 > 0 \text{ and } e_r = 0 \text{ if } g_3 = 0. \quad (3.26)$$

Motion is possible for $\rho \geq e_r$.

Absorbed paths exist if the range $e_r \leq \rho < \infty$ includes $r_{\text{sing}} < r < \infty$. In that case, we will have $e_r \leq \rho_\infty < \rho_{\text{sing}}$ and the solution will be given by (3.12) where the upper limit of integration “ ∞ ” is replaced by “ ρ_{sing} ”.

If $\rho_\infty < e_r < \rho_{\text{sing}}$, the solution is a trapped path given by (3.11).

One can envisage other situations as $\rho_{\text{sing}} < e_r$ and so on. However, we are giving examples that are more or less related to EMD and Reissner-Nordström black holes.

The case $g_2 = g_3 = 0$ implies $e_r = 0$ (this is the only case where the three real roots of $w(\rho) = 0$ are equal). This is no difference from the two cases discussed above for the generic case. However, if $\rho(r)$ were an increasing function of r and $\rho_\infty > e_r = 0$, the angle Θ would diverge as

$$\Theta - C = \int_{\rho}^{e_r=0} \frac{d\rho'}{\sqrt{4\rho'^3}} \propto \lim_{\rho' \rightarrow 0^+} \frac{1}{\sqrt{\rho'}}$$

for paths approaching $e_r = 0$ from the right. This is not a logarithmic behavior as the one we have seen earlier. These spiral paths would approach, without reaching it, the unstable circular path at $\rho = e_r = 0$.

4 Phantom and Normal Reissner-Nordström black holes

It is difficult or impossible to reduce (2.13) to (1.2) for any γ . For $\gamma = \pm 2$ and probably for some other values, the problem can be tackled semi-analytically in a similar way to what is done in [4, 6, 16]: Limits to the analytical treatment are 1) lack of “compact” solutions to the polynomial equation (or its polynomial reduced form) $P(r) = 0$ and/or 2) lack of solutions to the generally non-polynomial equation $\Delta \equiv g_2^3 - 27g_3^2 = 0$. All that said does not

apply to the cases $\gamma = 1$ and $\gamma = 0$, which can be entirely analytically solved. In this section we investigate the former case and in the next one we tackle the latter case.

Setting $\gamma = 1$ in (2.6, 2.7, 2.9, 2.13) with $\varepsilon = 0$ we obtain

$$M = \frac{r_+ + r_-}{2}, \quad q = \pm \sqrt{\eta_2 r_- r_+}, \quad r_- = \eta_2 |r_-|, \quad r_- r_+ = \eta_2 q^2 \quad (4.1)$$

$$r_{\pm} = M \pm \sqrt{M^2 - \eta_2 q^2} = M(1 \pm \sqrt{1 - \eta_2 a^2}) \quad (4.2)$$

$$\left(\frac{du}{d\phi} \right)^2 = \ell - u^2 + 2Mu^3 - \eta_2 q^2 u^4 \quad (4.3)$$

where $\ell = E^2/L^2 = 1/b^2 \geq 0$.

Let $u_r = 1/r_r$ be *any* real root of the polynomial in the r.h.s of (4.3):

$$\ell - u_r^2 + 2Mu_r^3 - \eta_2 q^2 u_r^4 = 0 \quad [\ell = \eta_2 q^2 u_r^2 (u_r - u_-)(u_r - u_+) \geq 0]. \quad (4.4)$$

Since $\ell \geq 0$, *all* the real roots of the polynomial are either greater than $u_- > u_+ > 0$ or smaller than u_+ for $\eta_2 = +1$ and they are *all* between $u_- < 0$ and $u_+ > 0$ for $\eta_2 = -1$ (Figure 9).

If $\ell \neq 0$ ($\ell > 0$), we have necessarily $u_r \neq u_+$, $u_r \neq u_-$ and $u_r \neq 0$: It is not possible to find solutions with $\ell \neq 0$ and $r_r = r_+$ [19, Eqs. (65, 66)].

If u_r is a root with multiplicity 1, following the general procedure, we introduce the radial coordinate $y = u - u_r$ followed by $z = 1/y$ and finally

$$\rho = \frac{3C_1 z + C_2}{12} = \frac{C_1}{4(u - u_r)} + \frac{C_2}{12} = -\frac{C_1}{4} \frac{r_r r}{r - r_r} + \frac{C_2}{12}, \quad \Theta = \phi \quad (4.5)$$

with

$$C_1 \equiv 2u_r(3Mu_r - 1 - 2\eta_2 q^2 u_r^2) = 2u_r(1 - Mu_r) - 4\ell/u_r \quad (4.6)$$

$$C_2 \equiv 6Mu_r - 1 - 6\eta_2 q^2 u_r^2 = 5 - 6Mu_r - 6\ell/u_r^2. \quad (4.7)$$

Eqs. (4.5) reduce (4.3) to (1.2) with

$$g_2 = \frac{1}{12} - \ell Q^2, \quad (Q^2 = \eta_2 q^2) \quad (4.8)$$

$$g_3 = \frac{1 - 54\ell M^2 + 36\ell Q^2}{216} \quad (4.9)$$

$$\begin{aligned} \Delta &= \ell[M^2(1 + 36\ell Q^2) - 27\ell M^4 - Q^2(1 + 4\ell Q^2)^2]/16 \\ &= -\ell[16Q^6\ell^2 + (27M^4 - 36M^2Q^2 + 8Q^4)\ell + Q^2 - M^2]/16 \\ &= -Q^6\ell(\ell - \ell_-)(\ell - \ell_+) = -\eta_2 q^6\ell(\ell - \ell_-)(\ell - \ell_+) \end{aligned} \quad (4.10)$$

where we have used (4.4) to eliminate u_r from the expressions of g_2, g_3 . The new parameters (ℓ_-, ℓ_+) are defined

by

$$q^2 \ell_{\pm} = \eta_2 \frac{-27 + 36\eta_2 a^2 - 8a^4 \pm (9 - 8\eta_2 a^2)^{3/2}}{32a^4}. \quad (4.11)$$

In the physical case $a^2 = q^2/M^2 < 1$, $0 < \ell_+ < \ell_-$ for the phantom black hole and $\ell_- < 0 < \ell_+$ for the normal one.

The transformation (4.5) “splits” the point $r = r_r$ into r_r^+ and r_r^- (corresponding to u_r^- and u_r^+ , respectively). If $C_1 < 0$ then r_r^+ and r_r^- are sent to $\rho = \rho_r = +\infty$ and $\rho = -\infty$, respectively, and if $C_1 > 0$ the latter limits are reversed. As we shall see in the Appendix, it is always possible to choose the real root u_r so that $C_1 < 0$: choose u_r to be the smallest root or the largest one for the phantom or normal black hole solution, respectively. The points ρ_{∞} , ρ_0 and ρ_{\pm} (corresponding to $r = +\infty$, $r = 0$ and $r = r_{\pm}$) on the ρ -axis are given by

$$\rho_{\infty} = \frac{\ell}{2u_r^2} - \frac{1}{12}, \quad \rho_0 = \frac{C_2}{12} = \frac{5}{12} - \frac{\ell}{2u_r^2} - \frac{Mu_r}{2} \quad (4.12)$$

$$\rho_{\pm} = \frac{C_1}{4(u_{\pm} - u_r)} + \frac{C_2}{12} \quad (4.13)$$

which depend on u_r whose analytic expression in terms of (M, q^2, ℓ) is sizable.

4.1 Three distinct real roots for $w(\rho) = 0$

For the phantom case, we derive in the Appendix (Eq. (A.2)) the following order relations for the ρ -parameters

$$e_3 < \rho_{\infty} < e_2 < e_1 < \rho_+ < \rho_0 < \rho_- < \rho_r = +\infty. \quad (4.14)$$

The only possible paths are scattering ones from ρ_{∞} to e_2 given by (3.9, 4.5) or trapped ones from any point $e_1 \leq \rho \leq \rho_+$ to the singularity at ρ_0 [from any point $r_+ \leq r \leq r_1 = 1/u_1$, where $u_1 < u_+$ is the largest root of $\ell - u^2 + 2Mu^3 + q^2u^4 = 0$ (Figure 9 (a)), to $r = 0$]. If the trapped path starts from $\rho = e_1$ its equation is given by (3.11, 4.5).

For the normal case, Eq. (A.6) of the Appendix reads

$$\rho_0 < e_3 < \rho_{\infty} < e_2 < e_1 < \rho_+ < \rho_- < \rho_r = +\infty. \quad (4.15)$$

The only possible paths are scattering ones from ρ_{∞} to e_2 given by (3.9, 4.5) or many-world ones from any point $e_1 \leq \rho \leq \rho_+$ to $\rho = +\infty$ [from any point $r_+ \leq r \leq r_1 = 1/u_1$ to $r_r = 1/u_r$ where $u_r > u_-$ is the largest root of $\ell - u^2 + 2Mu^3 - q^2u^4 = 0$ and $u_1 < u_+$ is the second largest root (Figure 9 (b))]. If the many-world path starts from $\rho = e_1$ its equation is given by (3.11, 4.5).

4.2 Two distinct real roots for $w(\rho) = 0$

This corresponds to (Eq. (3.13))

$$\frac{1}{12} - \ell\eta_2 q^2 > 0 \quad \text{and} \quad (\ell = 0, \ell = \ell_- \quad \text{or} \quad \ell = \ell_+). \quad (4.16)$$

4.2.1 Case $\ell = 0$.

As we do and explain in the Appendix we choose $u_r = u_-$, which is the smallest (respectively largest) root for the phantom (respectively normal) solution when $\ell = 0$. In this case $C_1 < 0$ and $\rho(r)$ is a decreasing function of r [as $r \rightarrow r_-^+$ (from the right), $\rho \rightarrow +\infty$]. The order relations as given in (A.3, A.7) read

$$\eta_2 = -1 : e_3 = e_2 = \rho_\infty = -\frac{1}{12} < e_1 = \rho_+ = \frac{1}{6} < \rho_0 < \rho_- = +\infty \quad (4.17)$$

$$\eta_2 = +1 : \rho_0 < e_3 = e_2 = \rho_\infty = -\frac{1}{12} < e_1 = \rho_+ = \frac{1}{6} < \rho_- = +\infty. \quad (4.18)$$

Since ρ_0 is a singularity for the phantom black hole, there is a trapped path for this hole from $\rho_+ \rightarrow \rho_0$ given by (3.18, 3.19) with $\eta_2 = -1$.

There is a many-world periodic path for the normal black hole from $\rho_+ \rightarrow \rho_-$ given by (3.17, 4.5). Using (4.5) with $u = 1/r$, $u_r = u_-$, $C_1 = -(r_+ - r_-)/r_-^2$ and $C_2 = (2r_- - 3r_+)/r_-$ we have

$$4(\rho - e_1) = \frac{r_+ - r}{r - r_-} = \frac{(r_+ - r)^2}{2Mr - \eta_2 q^2 - r^2} \quad (\eta_2 = +1) \quad (4.19)$$

then using the first equation (3.17) with $k = \sqrt{e_1 - e_3} = 1/2$ leads to (3.18, 3.19).

Had we chosen $u_r = u_+$, instead of $u_r = u_-$, we would reach the same conclusions concerning the nature of the paths. In this case, we would have $C_1 = 2u_+(1 - Mu_+) > 0$, $\rho(r)$ is an increasing function of r [as $r \rightarrow r_+^-$ (from the left), $\rho \rightarrow +\infty$] and

$$\eta_2 = -1 : e_3 = e_2 = \rho_\infty = -\frac{1}{12} < e_1 = \rho_- = \frac{1}{6} < \rho_0 < \rho_+ = +\infty \quad (4.20)$$

$$\eta_2 = +1 : e_3 = e_2 = \rho_\infty = -\frac{1}{12} < \rho_0 < e_1 = \rho_- = \frac{1}{6} < \rho_+ = +\infty. \quad (4.21)$$

But instead of (4.19) we would obtain

$$4(\rho - e_1) = \frac{r - r_-}{r_+ - r} = \frac{2Mr - \eta_2 q^2 - r^2}{(r_+ - r)^2} \quad \text{and} \quad \tan \left[\frac{\phi - C}{2} \right] = \sqrt{\frac{r - r_-}{r_+ - r}}. \quad (4.22)$$

4.2.2 Cases $\ell = \ell_\pm$.

Now we consider the cases $\ell = \ell_\pm$. Note that in this case g_2, g_3 are, by (4.11), functions of (a^2, η_2) only

$$g_2 = \frac{1}{12} - \eta_2(q^2 \ell_\pm)$$

$$g_3 = \frac{1 - (54/a^2)(q^2 \ell_\pm) + \eta_2 36(q^2 \ell_\pm)}{216}$$

and that ℓ_\pm are functions of (M, a^2, η_2)

$$\ell_\pm = \eta_2 \frac{-27 + 36\eta_2 a^2 - 8a^4 \pm (9 - 8\eta_2 a^2)^{3/2}}{32M^2 a^6}$$

leading

$$\lim_{a^2 \rightarrow 0} \ell_- = +\infty \ (\eta_2 = -1) \text{ and } \lim_{a^2 \rightarrow 0} \ell_+ = \frac{1}{27M^2} \ (\eta_2 = \pm 1).$$

With $b = \sqrt{1/\ell}$, the last two limits are the Schwarzschild limit for the impact parameter ($3\sqrt{3}M$) allowing photons to orbit endlessly the hole around the photon sphere without reaching it.

$\ell = \ell_+$. If $\ell = \ell_+$ and $\eta_2 = -1$, $g_2 > 0$ and $g_3 < 0$. A similar case has been treated in Eqs. (3.20) to (3.23). There is a root with multiplicity 2 at $\rho = e_1 = e_2 = (1/2)\sqrt{g_2/3}$. The corresponding root $u = u_1$ is such that the r.h.s of (4.3) reads: $\ell_+ - u^2 + 2Mu^3 + q^2u^4 = q^2(u - u_1)^2(u - u_r)(u - u_3)$ where $u = u_3$ corresponds to $\rho = e_3 = -2e_1$. This is the case where the point M_1 is on the u -axis (Figure 9 (a)). The order relations are given in (A.4):

$$e_3 < \rho_\infty < e_1 = e_2 < \rho_+ < \rho_0 < \rho_- < \rho_r = +\infty. \quad (4.23)$$

There is an unstable circular path at

$$\rho = e_1 = \frac{\sqrt{(9+8a^2)(9+4a^2-3\sqrt{9+8a^2})}}{24\sqrt{2}a^2}$$

corresponding to $r = r_1 = 1/u_1$ (the photon sphere) with⁶

$$r_1 = \frac{\sqrt{9+8a^2}+3}{2}M > 3M > r_+ \quad (4.24)$$

and spiral paths from $r = +\infty$ ($\rho = \rho_\infty$) to $r = r_1$ ($\rho = e_1$) and from $r = r_+$ ($\rho = \rho_+$) to $r = r_1$ ($\rho = e_1$) given by (3.23) and (4.5) with $k = \sqrt{3e_1}$. There is also a trapped path from $r = r_1$ ($\rho = e_1$) to $r = 0$ ($\rho = \rho_0$) given by (3.23) with the + sign. In the limit $a^2 \rightarrow 0$, $r_1 \rightarrow 3M$ which is the Schwarzschild limit.

If $\ell = \ell_+$ and $\eta_2 = +1$, $g_2 > 0$ and $g_3 < 0$. There is a root with multiplicity 2 at $\rho = e_1 = e_2 = (1/2)\sqrt{g_2/3}$. The corresponding root $u = u_1$ is such that the r.h.s of (4.3) reads: $\ell_+ - u^2 + 2Mu^3 - q^2u^4 = q^2(u - u_1)^2(u - u_r)(u - u_3)$ where $u = u_3$ corresponds to $\rho = e_3 = -2e_1$. This is the case where the point M_3 is on the u -axis (Figure 9 (b)). The order relations are given in (A.8):

$$\rho_0 < e_3 < \rho_\infty < e_1 = e_2 < \rho_+ < \rho_- < \rho_r = +\infty. \quad (4.25)$$

There is an unstable circular path at

$$\rho = e_1 = \frac{\sqrt{(8a^2-9)(4a^2-9+3\sqrt{9-8a^2})}}{24\sqrt{2}a^2}$$

⁶ u_1 is the largest root of $\ell_+ - u^2 + 2Mu^3 + q^2u^4 = 0$ when M_1 is on the u -axis (Figure 9 (a)).

corresponding to $r = r_1 = 1/u_1$ (the photon sphere) with⁷

$$r_+ < r_1 = \frac{\sqrt{9-8a^2}+3}{2}M < 3M \quad (4.26)$$

and spiral paths from $r = +\infty$ ($\rho = \rho_\infty$) to $r = r_1$ ($\rho = e_1$) and from $r = r_+$ ($\rho = \rho_+$) to $r = r_1$ ($\rho = e_1$) given by (3.23) and (4.5) with $k = \sqrt{3e_1}$. There is also a many-world periodic bound path from $r > r_+$ through $r = r_-$ to $r = r_r > 0$ which emerges in another copy of the space-time after crossing $r = r_-$. This is also given (3.23) with the $+$ sign. In the limit $a^2 \rightarrow 0$, $r_1 \rightarrow 3M$ which is the Schwarzschild limit.

$\ell = \ell_-$. We have necessarily $\eta_2 = -1$ since $\ell_- < 0$ for normal Reissner-Nordström black hole. In this case, the r.h.s of (4.3), $\ell_- - u^2 + 2Mu^3 + q^2u^4$, has only one real root⁸ $u_1 = -[\sqrt{9+8a^2}+3]/(4a^2M) < 0$ with multiplicity 2 and two complex roots⁹. Thus, the r.h.s of (4.3) is always positive with only absorbed paths from spatial infinity to the singularity at $r = 0$ given by (3.12) and (4.5).

4.3 One real root for $w(\rho) = 0$

This corresponds to (Eq. (3.24))

$$\frac{27+36a^2+8a^4-(9+8a^2)^{3/2}}{32M^2a^6} < \ell < \frac{27+36a^2+8a^4+(9+8a^2)^{3/2}}{32M^2a^6} \quad (\eta_2 = -1) \quad (4.27)$$

$$\ell > \frac{36a^2-27-8a^4+(9-8a^2)^{3/2}}{32M^2a^6} \quad (\eta_2 = +1). \quad (4.28)$$

For the phantom solution ($\eta_2 = -1$), this is the case where the point M_1 is above the u -axis and M_3 is below it (Figure 9 (a)). The two real roots ($u_r < u_3$) of $\ell - u^2 + 2Mu^3 + q^2u^4 = 0$ are negative and (u_1, u_2) are now complex roots, so (e_1, e_2) no longer exist. Eqs. (A.1, A.2) become

$$\begin{aligned} u_- < u_r < u_3 < 0 < u_+ \\ e_3 < \rho_\infty < \rho_+ < \rho_0 < \rho_- < \rho_r = +\infty \end{aligned} \quad (4.29)$$

with only absorbed paths from spatial infinity to the singularity at $r = 0$ given by (3.12) and (4.5).

For the normal solution ($\eta_2 = +1$), this is the case where the point M_3 is above the u -axis (Figure 9 (b)). The two real roots of $\ell - u^2 + 2Mu^3 - q^2u^4 = 0$ satisfy $u_3 < 0$ and $u_r > u_- > 0$ is the largest one. (u_1, u_2) are now complex roots, so (e_1, e_2) no longer exist. Eqs. (A.5, A.6) become

$$\begin{aligned} u_3 < 0 < u_+ < u_- < u_r \\ \rho_0 < e_3 < \rho_\infty < \rho_+ < \rho_- < \rho_r = +\infty. \end{aligned} \quad (4.30)$$

⁷ u_1 is smallest positive root of $\ell_+ - u^2 + 2Mu^3 - q^2u^4 = 0$ when the point M_3 is on the u -axis (Figure 9 (b)).

⁸ u_1 is the only real root of $\ell_- - u^2 + 2Mu^3 + q^2u^4 = 0$ when the point M_3 is on the u -axis (Figure 9 (a)).

⁹In this case to reduction of (4.3) does not lead to (1.2); rather, it leads to a similar equation with an irreducible quadratic form on the r.h.s., a polynomial of degree 2 with complex roots.

The only existing paths are two-world scattering paths from spatial infinity to $r = r_r = 1/u_r > 0$ given by (3.12) and (4.5).

5 Null geodesics of phantom and normal EMD

In section we restrict ourselves to the case $\gamma = 0$ which corresponds to $\eta_1 = +1$, then (2.3) implies $\eta_2 = -1$ for the cosh solution and $\eta_2 = +1$ for the sinh one. Thus we will be considering E-anti-MD for the cosh solution and normal EMD for the sinh one.

Instead of $u = (1/r_-) - (f_-/r_-)$, we use f_- as a radial coordinate. This way we reduce (2.13) for light paths ($\varepsilon = 0$) to

$$\left(\frac{df_-}{d\phi}\right)^2 = [\alpha f_-^3 - (3\alpha + 1)f_-^2 + (3\alpha + \beta + 2)f_- - (\alpha + 1)]f_- \quad (5.1)$$

where, using (2.8),

$$\alpha \equiv -\frac{r_+}{r_-} = -\eta_2 \frac{2M^2}{q^2} = -\eta_2 \frac{2}{a^2}, \quad \beta \equiv \frac{r_-^2 E^2}{L^2} = \frac{q^4 E^2}{M^2 L^2} = \frac{q^4}{M^2 b^2} \geq 0. \quad (5.2)$$

In the physical case $a^2 > 1$, to which we restrict ourselves, α is constrained by

$$\alpha > 2 \text{ if } \eta_2 = -1, \quad \alpha < -2 \text{ if } \eta_2 = +1 \quad (5.3)$$

for the phantom cosh and normal sinh solutions, respectively.

The next step is to introduce the variable $R = 1/(f_- - f_0)$ where f_0 is a zero of the fourth order polynomial in f_- on the r.h.s of (5.1). We choose $f_0 = 0$, leading to

$$\left(\frac{dR}{d\phi}\right)^2 = \alpha - (3\alpha + 1)R + (3\alpha + \beta + 2)R^2 - (\alpha + 1)R^3.$$

The final steps consist in eliminating the term in R^2 and rescaling ϕ by introducing the Weierstrass coordinates (ρ, Θ) defined by

$$R = -\frac{4^{1/3}}{(\alpha + 1)^{1/3}}\rho + \frac{3\alpha + \beta + 2}{3(\alpha + 1)} \quad (5.4)$$

$$d\Theta = -\eta_2 \frac{(\alpha + 1)^{1/3}}{4^{1/3}} d\phi, \quad (d\phi \cdot d\Theta > 0) \quad (5.5)$$

and $d\phi \cdot d\Theta > 0$ by (5.3). The reduced equation is (1.2): $(d\rho/d\Theta)^2 = 4\rho^3 - g_2\rho - g_3$ with

$$g_2 = \frac{4^{1/3}}{3} \frac{1 + 2(2 + 3\alpha)\beta + \beta^2}{(1 + \alpha)^{4/3}} \quad (5.6)$$

$$g_3 = \frac{2 - 3(5 + 12\alpha + 9\alpha^2)\beta - 6(2 + 3\alpha)\beta^2 - 2\beta^3}{27(1 + \alpha)^2}. \quad (5.7)$$

Note that, if α is restricted by (5.3), $\rho(r)$ is a decreasing function of r for all η_2 . $\rho(r)$ and its inverse function are given by

$$\rho = \frac{(\beta - 1)r - (3\alpha + \beta + 2)r_-}{3 \cdot 4^{1/3}(1 + \alpha)^{2/3}(r - r_-)} \quad \text{and} \quad r = \frac{r_-[3\alpha + \beta + 2 - 3 \cdot 4^{1/3}(1 + \alpha)^{2/3}\rho]}{\beta - 1 - 3 \cdot 4^{1/3}(1 + \alpha)^{2/3}\rho} \quad (5.8)$$

so that, using $r_- = \eta_2|r_-|$ and $\alpha + 1 = -\eta_2|\alpha + 1|$, we arrive at $d\rho/dr = -|r_-||\alpha + 1|^{1/3}/[4^{1/3}(r - r_-)^2]$.

In the limit $r \rightarrow r_-$, $\rho \rightarrow -3r_-(\alpha + 1)/(r - r_-) = 3|r_-||\alpha + 1|/(r - r_-)$ for all η_2 . Thus the transformation (5.8) “splits” the point r_- into r_-^- and r_-^+ and sends the point r_-^- to $\rho = -\infty$ and the point r_-^+ to $\rho = \rho_- = +\infty$. The points ρ_∞ , ρ_0 and ρ_+ (corresponding to $r = +\infty$, $r = 0$ and $r = r_+$) on the ρ -axis are given by

$$\rho_\infty = \frac{\beta - 1}{3 \cdot 4^{1/3}(\alpha + 1)^{2/3}}, \quad \rho_0 = \frac{3\alpha + \beta + 2}{3 \cdot 4^{1/3}(\alpha + 1)^{2/3}}, \quad \rho_+ = \frac{\beta + 2}{3 \cdot 4^{1/3}(1 + \alpha)^{2/3}} \quad (5.9)$$

and $\rho_0 > 0$ for phantom black holes. If (e_1, e_2, e_3) are real, the order relations of these roots with respect to $(\rho_\infty, \rho_0, \rho_+)$ depend on $(\alpha, \beta) = \vec{p}$. This will be done for each case (phantom or normal) separately.

Ordering $(\rho_\infty, \rho_0, \rho_+)$ is also done separately as follows. For the phantom cosh black hole we have $r_-^+ < 0 < r_+ < +\infty$ which leads to $(\rho(r))$ is always decreasing

$$\rho_\infty < \rho_+ < \rho_0 < \rho_- = +\infty. \quad (5.10)$$

For the normal sinh black hole we have $0 < r_-^- < r_-^+ < r_+ < +\infty$. But since $\rho(r)$ is always decreasing, if one moves on the r -axis along the path $r = +\infty \rightarrow r_+ \rightarrow r_-^+ \rightarrow r_-^- \rightarrow 0$, the corresponding point on the ρ -axis moves along the path $\rho_\infty \rightarrow \rho_+ \rightarrow \rho_- = +\infty \rightarrow$ (in a circular rotation) $-\infty \rightarrow \rho_0$. Thus

$$\rho_0 < \rho_\infty < \rho_+ < \rho_- = +\infty. \quad (5.11)$$

Let $(\beta_1, \beta_2, \beta_3, \beta_4)$ be the following α -functions

$$\beta_{1,2} = -(2 + 3\alpha) \mp \sqrt{3(1 + \alpha)(1 + 3\alpha)} \quad (1 \rightarrow -, 2 \rightarrow +) \quad (5.12)$$

$$\beta_{3,4} = \frac{1 - 18\alpha - 27\alpha^2 \pm (1 + 9\alpha)\sqrt{(1 + \alpha)(1 + 9\alpha)}}{8\alpha} \quad (3 \rightarrow +, 4 \rightarrow -) \quad (5.13)$$

in terms of which we have

$$g_2 = \frac{4^{1/3}}{3} \frac{(\beta - \beta_1)(\beta - \beta_2)}{(1 + \alpha)^{4/3}} \quad (5.14)$$

$$\Delta \equiv g_2^3 - 27g_3^2 = \frac{\beta[4 + (1 - 18\alpha - 27\alpha^2)\beta - 4\alpha\beta^2]}{(1 + \alpha)^2} = \frac{-4\alpha\beta(\beta - \beta_3)(\beta - \beta_4)}{(1 + \alpha)^2}.$$

5.1 The phantom cosh black hole: $\alpha > 2$, $\eta_2 = -1$

In this case $g_2 > 0$ for all $\beta \geq 0$ (Eq. (5.2)), $\beta_4 < 0$ and $\beta_3 > 0$.

5.1.1 Three distinct real roots for $w(\rho) = 0$

This corresponds to (Eq. (3.3))

$$0 < \beta < \beta_3 \quad (5.15)$$

which leads, using (3.4, 5.9, 5.10), to

$$e_3 < \rho_\infty < e_2 < e_1 < \rho_+ < \rho_0 < \rho_- = +\infty \quad (\rho_\infty < 0). \quad (5.16)$$

To order $(\rho_\infty, \rho_0, \rho_+)$ with respect to (e_1, e_2, e_3) we may use different methods as plotting the surfaces $\rho_+ - e_1$ and so on or simply evaluate the Weierstrass polynomial and its derivatives $w' = 12\rho^2 - g_2$ and $w'' = 24\rho$ at $(\rho_\infty, \rho_0, \rho_+)$. For instance, $w(\rho_+) > 0$, $w'(\rho_+) > 0$ and $w''(\rho_+) > 0$.

This case has been treated in subsubsection 3.1.1 case (b). Since ρ_0 is a singularity for the cosh black hole, there is a trapped path from e_1 to ρ_0 given by (3.11, 5.5, 5.8). The scattering path from $\rho_\infty \rightarrow e_2 \rightarrow \rho_\infty$ is given by (3.9, 5.5, 5.8).

5.1.2 Two distinct real roots for $w(\rho) = 0$

This corresponds to (Eq. (3.13))

$$\beta = 0 \text{ or } \beta = \beta_3. \quad (5.17)$$

For $\beta = 0$, $\rho_+ = e_1$ and $g_3 > 0$ so that $g_3 = (g_2/3)\sqrt{g_2/3}$. The relations (5.16) are still valid, in the limit we have $e_3 = \rho_\infty = e_2$. This case has been treated in Eqs. (3.14) to (3.17). Since ρ_0 is a singularity for the cosh black hole, there is a trapped or terminating bound path from $\rho_+ = e_1$ ($r = r_+$) to ρ_0 ($r = 0$) given by (3.17, 5.5, 5.8) with $e_1 = -2e_3 = -2e_2 = -2\rho_\infty = 2^{1/3}/[3(1+\alpha)^{2/3}]$:

$$2^{2/3}(1+\alpha)^{2/3}\rho = \frac{2}{3} + \tan^2 \left[\frac{\Theta - C}{2^{1/3}(1+\alpha)^{1/3}} \right].$$

Substituting $\beta = 0$ into (5.6), then into (3.15) and the second equation (5.8) we obtain the radius of the stable circular path at $r = \infty$, as in the Schwarzschild case.

For $\beta = \beta_3$, $g_3 < 0$ so that $g_3 = -(g_2/3)\sqrt{g_2/3}$. This case has been treated in Eqs. (3.20) to (3.23). The relations (5.16) remain valid with $2e_1 = 2e_2 = -e_3 = \sqrt{g_2(\beta_3)/3}$ where the value of g_2 at $\beta = \beta_3$ given by

$$g_2(\beta_3) = \frac{(1+9\alpha)[1+9\alpha^2+\sqrt{A}+\alpha(2-3\sqrt{A})]}{48 \cdot 2^{1/3}\alpha^2(1+\alpha)^{1/3}}.$$

There are spiral paths given by (3.23, 5.5, 5.8) which approach the unstable circular path at $\rho = e_1$ from above(ρ_∞)/below(ρ_0). Using the above expression for $g_2(\beta_3)$ and substituting $\beta = \beta_3$ into (3.21) and the second equation (5.8) we obtain the radius r_1 of the unstable circular path where $A = (1+\alpha)(1+9\alpha)$ and $r_+ = 2M$

$$r_1 = \frac{8r_+(1+3\alpha)}{1 + \sqrt{A} - \alpha(10 + 27\alpha - 9\sqrt{A}) + \sqrt{2A}\sqrt{1 + \sqrt{A} + \alpha(2 + 9\alpha - 3\sqrt{A})}} \quad (\alpha > 2). \quad (5.18)$$

The limit $q^2 \rightarrow 0$ corresponds to $\alpha \rightarrow +\infty$. The radius r_1 , as given by (5.18), decreases from $(5 + \sqrt{57})r_+/8$ to the Schwarzschild limit $3r_+/2$ as α increases from $2 \rightarrow +\infty$.

5.1.3 One real root for $w(\rho) = 0$

This case corresponds to (Eq. (3.24))

$$\beta > \beta_3 \quad (5.19)$$

leading to

$$\rho_\infty < e_r < \rho_+ < \rho_0 < \rho_- = +\infty \quad (5.20)$$

where e_r is the real given by (3.25). For $\alpha > 2$ it is not possible to have $g_2 = 0$ and $g_3 = 0$, so there is no solution $e_r = 0$ with multiplicity 3. Since ρ_0 is a singularity for the cosh black hole, there is a trapped path from e_r to ρ_0 for the generic case $\beta > \beta_3$ given by (3.11, 5.5, 5.8).

5.2 The normal sinh black hole: $\alpha < -2$, $\eta_2 = +1$

In this case $0 < \beta_4 < \beta_1 < \beta_2 < \beta_3$. Thus, the condition $\Delta > 0$ (Eq. (5.14)) ensures $g_2 > 0$.

5.2.1 Three distinct real roots for $w(\rho) = 0$

This corresponds to (Eq. (3.3))

$$0 < \beta < \beta_4 \text{ or } \beta > \beta_3 \quad (5.21)$$

which leads, using (3.4, 5.9, 5.11), to

$$\rho_0 < e_3 < \rho_\infty < e_2 < e_1 < \rho_+ < \rho_- = +\infty \text{ if } 0 < \beta < \beta_4 \quad (\rho_\infty < 0) \quad (5.22)$$

$$e_3 < e_2 < \rho_0 < e_1 < \rho_\infty < \rho_+ < \rho_- = +\infty \text{ if } \beta > \beta_3. \quad (5.23)$$

For the case $0 < \beta < \beta_4$, which has been treated in subsubsection 3.1.1 case (d), the solution for the scattering path from $\rho_\infty \rightarrow e_2 \rightarrow \rho_\infty$ ($r = \infty \rightarrow r_2 \rightarrow r = \infty$) is given by (3.9, 5.5, 5.8). There is another path from $r = r_1$ ($\rho = e_1$) to $r = r_-$ ($\rho = \rho_-$), which is a trapped path for ρ_- is a null singularity for the sinh black hole. If we choose $\Theta = 0$ at $\rho = e_1$, then the solution is given by (3.11, 5.5, 5.8).

The case $\beta > \beta_3$ has been treated in subsubsection 3.1.1 case (c). Since ρ_- is a singularity, we have an absorbed path from spatial infinity to the singularity. The solution is given by (3.12, 5.5, 5.8).

5.2.2 Two distinct real roots for $w(\rho) = 0$

This corresponds to (Eq. (3.13))

$$\beta = 0, \beta = \beta_4 \text{ or } \beta = \beta_3. \quad (5.24)$$

The discussion of the case $\beta = 0$ for the normal sinh black hole is similar to that for the phantom cosh one. What was said in the first paragraph following (5.17) applies to this case if we replace “ ρ_0 ” by “ ρ_- ”, “ $r = 0$ ”

by “ $r = r_-$ ” and “cosh” by “sinh”. Thus, there a trapped path from ρ_+ to the singularity ρ_- given by (3.17, 5.5, 5.8).

For $\beta = \beta_4$, $g_3 < 0$ so that $g_3 = -(g_2/3)\sqrt{g_2/3}$. This case corresponds to the case $\beta = \beta_3$ of the phantom cosh black hole; the discussion in the second paragraph following (5.17) applies and the radius of the unstable circular path is obtained from (5.18) upon changing \sqrt{A} to $-\sqrt{A}$ [this is also true for $g_2(\beta_4)$ of this case which is derived from $g_2(\beta_3)$ of the phantom cosh black hole by $\sqrt{A} \rightarrow -\sqrt{A}$]

$$r_1 = \frac{8r_+(1+3\alpha)}{1 - \sqrt{A} - \alpha(10+27\alpha+9\sqrt{A}) - \sqrt{2A}\sqrt{1 - \sqrt{A} + \alpha(2+9\alpha+3\sqrt{A})}} \quad (\alpha < -2). \quad (5.25)$$

The limit $q^2 \rightarrow 0$ corresponds to $\alpha \rightarrow -\infty$. The radius r_1 , as given by (5.25), increases from $(7 + \sqrt{17})r_+/8 > r_+$ to the Schwarzschild limit $3r_+/2$ as $|\alpha|$ increases from $2 \rightarrow +\infty$ (α decreases from $-2 \rightarrow -\infty$).

For $\beta = \beta_3$, $g_3 > 0$ so that $g_3 = +(g_2/3)\sqrt{g_2/3}$. In this case $e_3 = e_2 = -2e_1$ (Eq. (3.14)) and all remaining inequalities in (5.23) are still valid. Since ρ_- is a singularity, we have an absorbed path from spatial infinity to the singularity. The solution is given by (3.12, 5.5, 5.8). The value $r_3 = r_2$ corresponding to $e_3 = e_2$, which should give the radius of the stable circular path, is such that $0 < r_3 = r_2 < r_- = r_{\text{sing}}$.

5.2.3 One real root for $w(\rho) = 0$

This case corresponds to (Eq. (3.24))

$$\beta_4 < \beta < \beta_3 \quad (5.26)$$

leading to

$$\rho_0 < e_r < \rho_\infty < \rho_+ < \rho_- = +\infty \quad (5.27)$$

where e_r is the real root given by (3.25). For $\alpha < -2$ it is not possible to have $g_2 = 0$ and $g_3 = 0$, so there is no solution $e_r = 0$ with multiplicity 3. In the generic case $\beta_1 \leq \beta \leq \beta_2$ and $g_3 \neq 0$ there is an absorbed path from ρ_∞ to the singularity at $\rho = \rho_- = +\infty$ given by (3.12, 5.5, 5.8).

6 Conclusion

To the first order of approximation, all black holes of phantom and normal EMD deflect light in the same manner. If we restrict ourselves to physical conditions [$a^2 \leq 1$ for $\eta_2 = -1$ and $a^2 \leq (1 + \gamma)/(2\gamma)$ for $\eta_2 = +1$], then 1) for γ larger than some γ_0 , which is likely in $(-0.2, 0)$ and depending on the parameters of the black hole, black holes of E-anti-M-(anti)-D (regardless the sign of η_1) theory cause light rays to deflect with larger angles than black holes of EM-(anti)-D do. The difference in the angles and the relative discrepancy ever increase with a^2 for fixed (u_n, γ) . For fixed (a^2, γ) , they increase with $1/r_n$ and diverge as r_n approaches the photon sphere of E-anti-M-(anti)-D black holes. 2) For $-1 < \gamma < \gamma_0$ light is more deflected by the black holes of EMD than by those of E-anti-MD, the relative discrepancy for larger values of the impact parameter is, however, much larger for the black holes of E-anti-MD.

Relativistic images [33] is another ingredient, besides deflection, allowing for the determination of the nature of matter. These observations happen on the photon sphere of the solution. If the latter is different from a Schwarzschild black hole, the images are observed for a radial distance larger or smaller than $3M$.

The method based on Weierstrass polynomial to integrate geodesic motion and determine exact solutions is equivalent to other methods using potential barriers and can be applied systematically. The advantage of using the method based on Weierstrass polynomial is that motion is allowed in at most two regions: in between the smallest root of the polynomial and the intermediate one and/or for values gritter than the largest root. This highly simplifies the problem. Some of the systematic resolutions consist in: 1) The angle of deflection has a standard formula for all problems that can be brought to Weierstrass differential equation. 2) If the two smallest and intermediate roots of the Weierstrass polynomial are equal for some value of the vector of parameters, there should be a stable circular path (photon sphere) for the corresponding radial coordinate r if the latter is within accessible limits to observers. 3) If the two largest and intermediate roots are equal for some value of the vector of parameters, there should be an unstable circular path for the corresponding radial coordinate r if the latter is within accessible limits to observers. 4) Existence of spiral paths, which approach endlessly the photon spheres, is a consequence of any of 1) or 2) or both. 5) Existence and identification of divergencies for the angle of deflection: a logarithmic one if 3) holds or a power law one (to the power $-1/2$) if the three real roots are zero. 6) Ordering of the parameters expressing spatial infinity, singularity, horizons and so on on the Weierstrass axis is derived by circular rotation (from their given order relations on the r -axis) in the one or the other way depending on the coordinate transformation relating the Weierstrass radial coordinate to the spherical radial one (increasing or decreasing).

Phantom Reissner-Nordström black holes are characterized by the existence of trapped and absorbed null paths which do not exist for normal Reissner-Nordström black holes. Their other non-common paths include many-world (periodic bound) and two-world paths which exist only for normal Reissner-Nordström black holes. Their common paths include scattering, spiral (existence of logarithmic divergencies) and unstable circular paths with radii approaching the Schwarzschild limit from above for phantom black holes and from below for normal ones.

Both phantom cosh and normal sinh black holes of EMD theory are characterized by the presence of scattering, trapped and unstable circular paths, thus spiral paths and existence of logarithmic divergencies. The photon spheres are larger or smaller than the Schwarzschild one, respectively, and approach it in the limit of no electric charge. The phantom solution has no absorbed path while the normal one does have.

Acknowledgments

Thanks are due to Gérard Clément (LAPTH) for helpful correspondence.

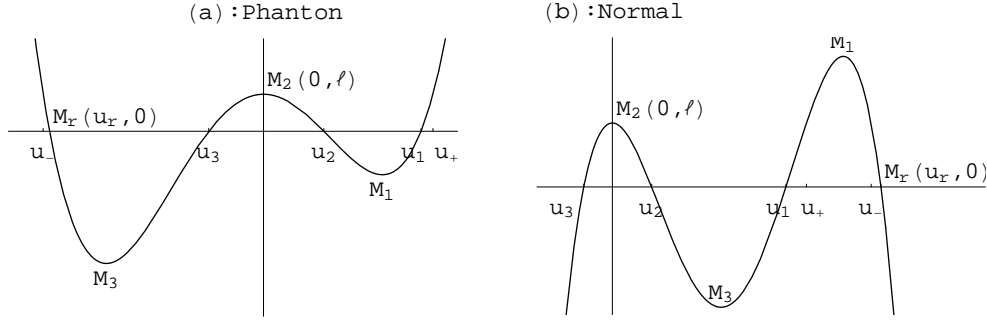


Figure 9: Plots of $Y = \ell - u^2 + 2Mu^3 - \eta_2 q^2 u^4$. (a): The phantom Reissner-Nordström black hole ($\eta_2 = -1$). u_r is the lowest root of $\ell - u^2 + 2Mu^3 + q^2 u^4 = 0$ with $u_- < u_r$. (b): The normal Reissner-Nordström black hole ($\eta_2 = +1$). u_r is the largest root of $\ell - u^2 + 2Mu^3 - q^2 u^4 = 0$ with $u_- < u_r$. For both plots, (u_+, u_-) are the intersections of the graphs of $Y = -u^2 + 2Mu^3 - \eta_2 q^2 u^4$ with the u -axis, which are the same graphs as those shown here but shifted downward ℓ units.

Appendix: Order relations for the phantom and normal Reissner-Nordström black holes

The phantom case $\eta_2 = -1$

In the case where all the four roots of $\ell - u^2 + 2Mu^3 + 2q^2 u^4 = 0$ have multiplicity 1 we can choose any root to perform the reduction of (4.3) to (1.2). For the phantom Reissner-Nordström black hole we choose u_r to be the smallest root as shown in Figure 9 (a)

$$u_- < u_r^- < u_r^+ < u_3 < 0 < u_2 < u_1 < u_+ < +\infty. \quad (\text{A.1})$$

As defined in the first expressions of Eqs. (4.6, 4.7), (C_1, C_2) are proportional to the first and second derivative of $\ell - u^2 + 2Mu^3 + q^2 u^4$ at $u = u_r$, respectively. At the point $M_r(u_r, 0)$ of the (a)-plot the function is decreasing and concave up (convex), thus $C_1 < 0$, $C_2 > 0$ and ρ is an increasing function of u (a decreasing function of r).

Since $C_1 < 0$, the coordinate transformation (4.5) “splits” the point u_r into u_r^+ and u_r^- , which correspond to $\rho = -\infty$ and $\rho = +\infty$, respectively. If one starts to move on the u -axis from the right to the left: from $u = +\infty$ ($r = 0$) to u_+ to u_1 to \dots to u_r^+ to u_r^- and finally to u_- . Since ρ is an increasing function of u , the corresponding point on the ρ -axis starts to move from ρ_0 to ρ_+ to e_1 to \dots to $\rho = -\infty$ then back in a circular rotation to $\rho = +\infty$ and finally to ρ_- . Thus, we have the following order relations for the ρ -parameters

$$-\infty < e_3 < \rho_\infty < e_2 < e_1 < \rho_+ < \rho_0 < \rho_- < \rho_r = +\infty. \quad (\text{A.2})$$

[Of course, this ordering can be derived by algebraic methods].

If $\ell = 0$ (M_2 on the u -axis), then $u_r = u_-$, $u_2 = u_3 = 0$ and $u_1 = u_+$ ($\rho_r = +\infty$, $e_2 = e_3$ and $e_1 = \rho_+$). Using (3.14, 4.6, 4.7, 4.8, 4.9, 4.12, 4.13) we obtain $12\rho_0 = 5 - [6\eta_2(1 + \sqrt{1 - \eta_2 a^2})/a^2]$ (with $\eta_2 = -1$) and the latter equation becomes

$$e_3 = e_2 = \rho_\infty = -\frac{1}{12} < e_1 = \rho_+ = \frac{1}{6} < \rho_0 < \rho_- = +\infty. \quad (\text{A.3})$$

If $\ell = \ell_+$, then $u_1 = u_2$ (M_1 on the u -axis) and (A.2) becomes

$$-\infty < e_3 < \rho_\infty < e_2 = e_1 < \rho_+ < \rho_0 < \rho_- < \rho_r = +\infty. \quad (\text{A.4})$$

If $\ell = \ell_-$, then $u_r = u_3$ (M_3 on the u -axis) and the root u_r has multiplicity 2.

The normal case $\eta_2 = +1$

In the case where all the four roots of $\ell - u^2 + 2Mu^3 - 2q^2u^4 = 0$ have multiplicity 1 we can choose any root to perform the reduction of (4.3) to (1.2). For the phantom Reissner-Nordström black hole we choose u_r to be the largest root as shown in Figure 9 (b)

$$u_3 < 0 < u_2 < u_1 < u_+ < u_- < u_r^- < u_r^+ < +\infty. \quad (\text{A.5})$$

At the point $M_r(u_r, 0)$ of the (b)-plot the function is decreasing and concave down (concave), thus $C_1 < 0$, $C_2 < 0$ and ρ is an increasing function of u (a decreasing function of r). Since $C_1 < 0$, the coordinate transformation (4.5) “splits” the point u_r into u_r^+ and u_r^- , which correspond to $\rho = -\infty$ and $\rho = +\infty$, respectively. If one starts to move on the u -axis from the right to the left: from $u = +\infty$ ($r = 0$) to u_r^+ to u_r^- to u_- to \dots to u_2 to u_3 and finally to $u = -\infty$. Since ρ is an increasing function of u , the corresponding point on the ρ -axis starts to move from ρ_0 to $\rho = -\infty$ then back in a circular rotation to $\rho = +\infty$ to ρ_- to \dots to e_2 to e_3 and finally to ρ_0 again. Thus, we have the following order relations for the ρ -parameters

$$-\infty < \rho_0 < e_3 < \rho_\infty < e_2 < e_1 < \rho_+ < \rho_- < \rho_r = +\infty. \quad (\text{A.6})$$

If $\ell = 0$ (M_2 on the u -axis), then $u_r = u_-$, $u_2 = u_3 = 0$ and $u_1 = u_+$ ($\rho_r = +\infty$, $e_2 = e_3$ and $e_1 = \rho_+$). Using (3.14, 4.6, 4.7, 4.8, 4.9, 4.12, 4.13) we obtain $12\rho_0 = 5 - [6\eta_2(1 + \sqrt{1 - \eta_2 a^2})/a^2]$ (with $\eta_2 = +1$) and the latter equation becomes

$$-\infty < \rho_0 < e_3 = e_2 = \rho_\infty = -\frac{1}{12} < e_1 = \rho_+ = \frac{1}{6} < \rho_- = +\infty \quad (\text{A.7})$$

If $\ell = \ell_+$, then $u_1 = u_2$ (M_3 on the u -axis) and (A.6) becomes

$$-\infty < \rho_0 < e_3 < \rho_\infty < e_2 = e_1 < \rho_+ < \rho_- < \rho_r = +\infty. \quad (\text{A.8})$$

If $\ell = \ell_- < 0$, there is still one real root $u_+ < u_r = u_1 < u_-$ (M_1 on the u -axis). This case is excluded, however, if it were possible for a photon to move with a negative energy, it could do it on a confined stable circle with radius $r_- < r = M(3 - \sqrt{9 - 8a^2})/2 < r_+$ which shrinks to zero as $a^2 = q^2/M^2$ approaches zero.

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